Rational RG flow, extension, and Witt class

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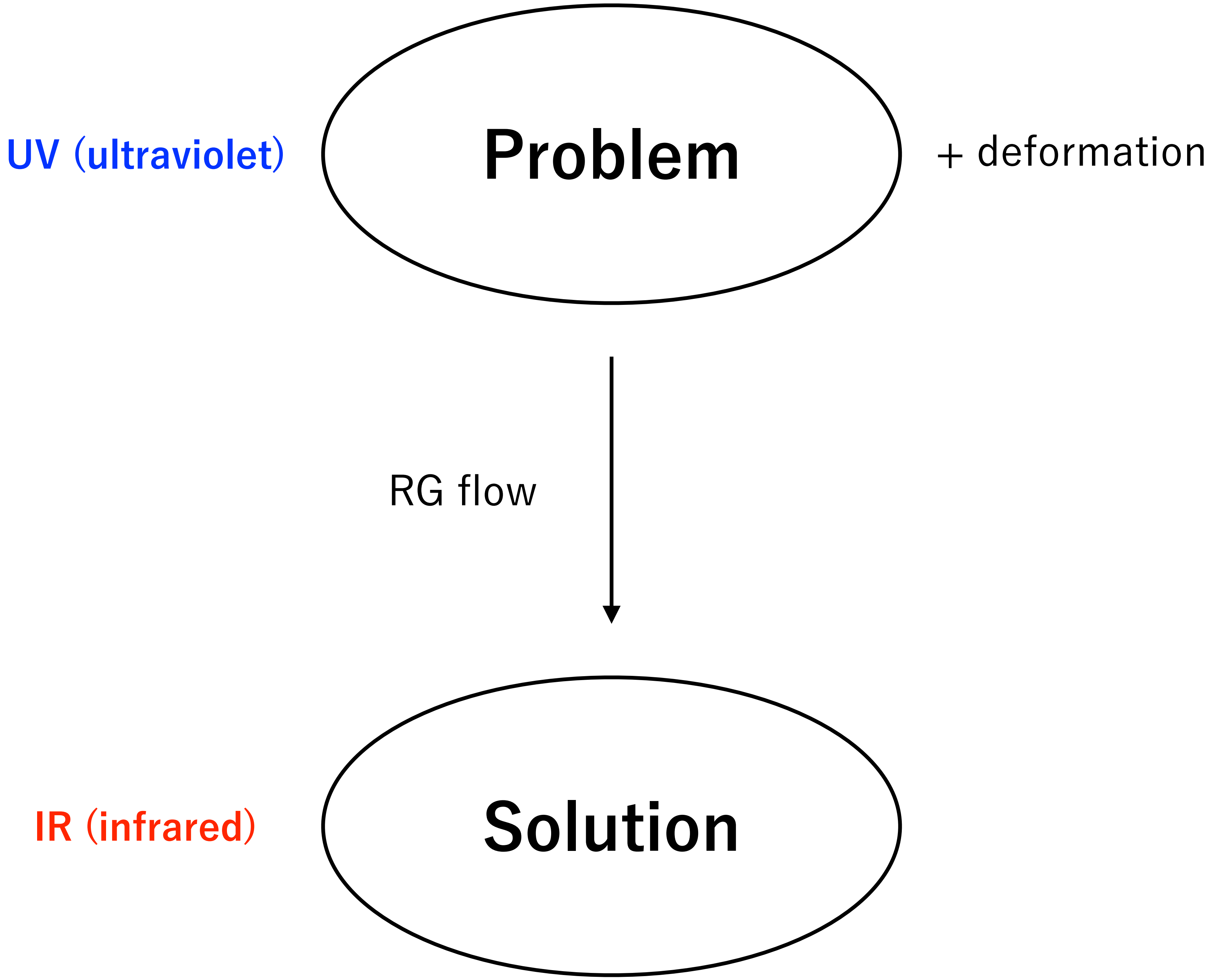
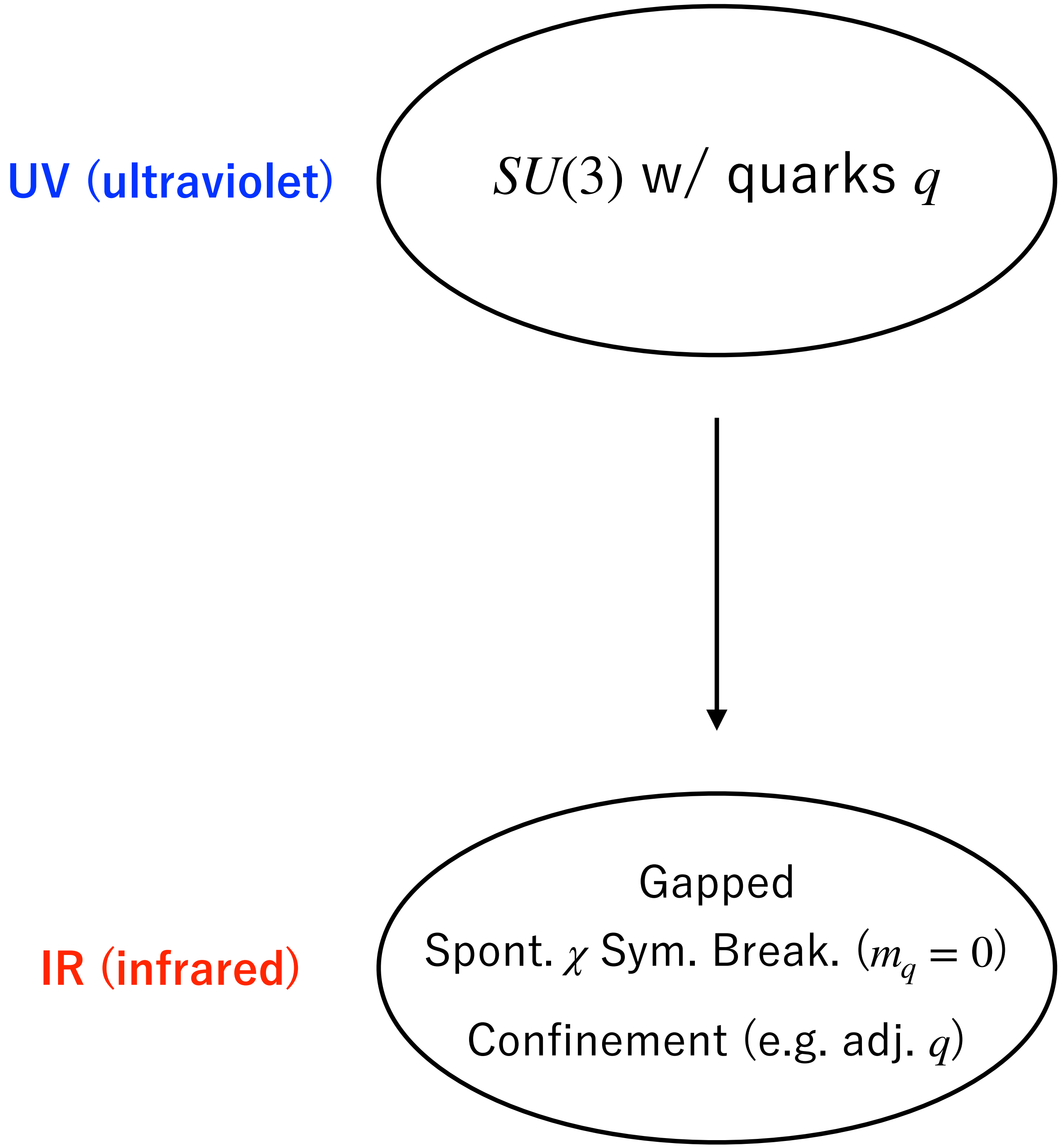
Based on
2412.08935 [hep-th] (KK)

Renormalization Group (RG) flow



Example: Quantum ChromoDynamics



Possible answers

Symmetry\Gap	Gapped (or TQFT)	Gapless (~CFT)
Preserved		
Spont. broken		

Example: QCD

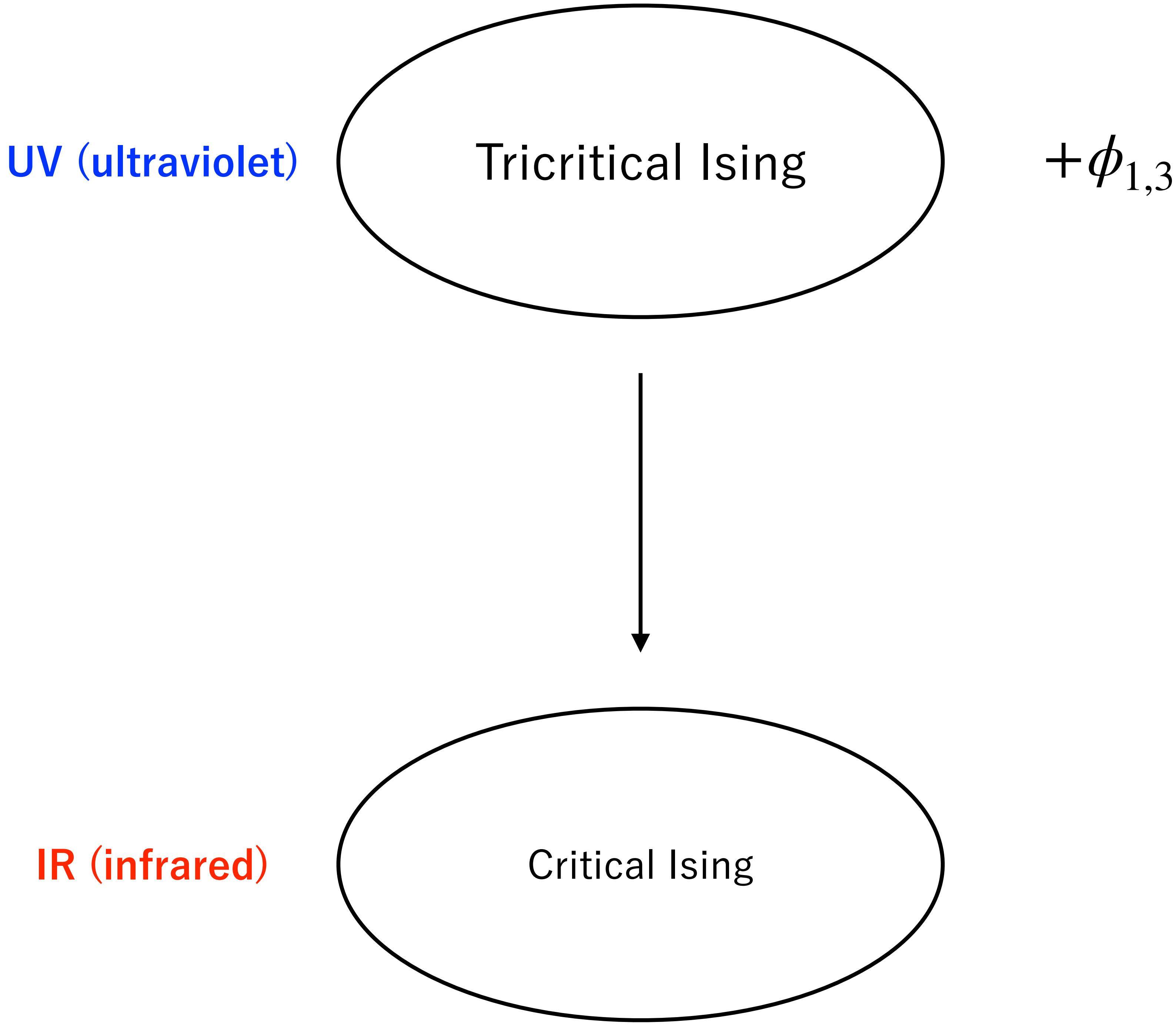
Symmetry\Gap	Gapped (or TQFT)	Gapless (~CFT)
Preserved	Confinement	
Spont. broken	SχSB	

Are **discrete** quantities invariant under **RG flows**?

No!

conformal dimension

Counterexample: Tricritical Ising+$\phi_{1,3}$ $\rightarrow$ Critical Ising



Counterexample: Tricritical Ising+$\phi_{1,3} \to$ Critical Ising

The relevant deformation preserves 3 symmetries $\{1, x, y\}$:

UV (ultraviolet)

$$\text{Tricritical Ising}: \quad 1_0 \quad x_{\frac{3}{2}} \quad y_{\frac{7}{16}}$$

$$\downarrow \quad \downarrow \quad \downarrow$$

IR (infrared)

$$\text{Critical Ising}: \quad 1_0 \quad X_{\frac{1}{2}} \quad Y_{\frac{1}{16}}$$

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UV (ultraviolet) $\text{Tricritical Ising}: \quad 1_0 \quad x_{\frac{3}{2}} \quad y_{\frac{7}{16}}$

$$\downarrow \quad \downarrow \quad \downarrow$$

IR (infrared) $\text{Critical Ising}: \quad 1_0 \quad X_{\frac{1}{2}} \quad Y_{\frac{1}{16}}$

While conformal dimensions (mod 1) are **discrete**,
they **jump** at conformal fixed point! $y_{\frac{7}{16}} \to Y_{\frac{1}{16}}$

- **Why** conformal dimensions jump?

- **What values** are allowed for h^{IR}?

- Can we use the jump to **solve RG flows**?

We answer all by proposing **mathematical definition** of (certain) **RG flow**.

Anticipated question

Then **why** ('t Hooft) anomalies match?

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An RG flow gives a **group homomorphism**

$$\phi : G_{UV} \to G_{IR}\,.$$

Its pullback **matches** anomalies α_{UV}, α_{IR} as

$$\alpha_{UV} = \phi^* \alpha_{IR}\,.$$

Content

1. **Math background**

2. **Definition**

3. **Examples**

Content

1. **Math background**
2. **Definition**
3. **Examples**

Modular fusion category

Generalized symmetry

= fusion category

Modular fusion category

fusion category

=analogue of **representation**

Modular fusion category

2-dim. irrep. of $SU(2)$ obeys

$$2 \otimes 2 = 4 = 1 \oplus 3.$$

Modular fusion category

Analogously, fusion category C has

- simple objects $c_i \in C$,
- fusion product $\otimes$,
- direct sum $\oplus$.

Modular fusion category

Example: Ising fusion category (FC)

Ising FC$=\{1, x, y\}$

i.e., rank=3

It has fusion product

$$x \otimes x = 1, \quad x \otimes y = y = y \otimes x, \quad y \otimes y = 1 \oplus x\,.$$

Modular fusion category

- Fusion category C may have **braiding** $c_{c_1,c_2} : c_1 \otimes c_2 \to c_2 \otimes c_1$.
  $(c_1, c_2 \in C)$

- **Braided fusion category** (BFC):=fusion category w/ braiding c.
  (w/ consistency conditions)

- **Pre-modular fusion category** (Pre-MFC):=spherical BFC w/ **the** quantum dimension d_i of $c_i \in C$.

Modular fusion category

・A Pre-MFC C is called **modular** or **modular fusion category** if the modular S-matrix

$$\widetilde{S}_{i,j} := tr(c_{j,i}c_{i,j})$$

$$= \sum_{c_k \in C} N_{ij}{}^{k} \frac{e^{2\pi i h_k}}{e^{2\pi i (h_i + h_j)}} d_k$$

is **non-singular**, i.e., $\det \widetilde{S} \neq 0$.

Modular fusion category

Example: Ising modular fusion category (MFC)

Ising MFC $=\{1, x, y\}$

$$x \otimes x = 1, \quad x \otimes y = y = y \otimes x, \quad y \otimes y = 1 \oplus x$$

$$\widetilde{S} = \begin{pmatrix} 1 & 1 & \sqrt{2} \\ 1 & 1 & -\sqrt{2} \\ \sqrt{2} & -\sqrt{2} & 0 \end{pmatrix}$$

(in basis $\{1, x, y\}$)

Content

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2. Definition

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Definition

(certain) **RG flow** w/ MFC C_{UV}

:=``simple fusion subcategory''

of $C_{UV} \boxtimes C_{IR}$

Definition

Just like groups, fusion categories can also be divided.

Group. Let G be a group and $N \subset G$ a normal subgroup.

$\Rightarrow$quotient subgroup G/N

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Group. Let G be a group and $N \subset G$ a normal subgroup.

$\Rightarrow$quotient subgroup G/N

Category. Let C be an MFC and $A \in C$ a ``normal subcat.'' .
(connected étale algebra)

$\Rightarrow$``quotient'' subcat. C/A
(category of local A-modules)

Definition

We introduce **equivalence relation** $C \sim C/A$.

The equivalence class $[C]$ is called the **Witt equivalence class**.

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[Davydov-Muger-Nikshych-Ostrik '10]
[Davydov-Nikshych-Ostrik '11]

$\forall [C]$, $\exists!$ ``simple'' representative $C' \in [C]$.

(completely anisotropic)

Definition

[2209.00016 (KK)]

An RG flow preserving a fusion category C_{UV} gives

a **monoidal functor** $F : C_{UV} \to C_{IR}$.

($\therefore C_{UV} = C_{IR}$ as fusion category)

Definition

[2209.00016 (KK)]

An RG flow is called **rational** if IR theory=RCFT.

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[2209.00016 (KK)]

An RG flow is called **rational** if IR theory=RCFT.

<u>Definition (conjecture).</u>

[2412.08935 (KK)]

$\{$Rational RG flow w/ MFC $C_{UV}\}$

$\cong \{$Completely anisotropic representative of $[C_{UV} \boxtimes C_{IR}]\}$

Definition

[2212.13851 (KK)]
[2412.08935 (KK)]

Theorem.

Let C_{UV} be the surviving MFC and C_{IR} be its image MFC in IR RCFT under monoidal functor $F : C_{UV} \to C_{IR}$. If the rational RG flow admits RG domain wall (a.k.a. RG interface), then surviving symmetry objects $c_j^{UV} \in C_{UV}, c_j^{IR} = F(c_j^{UV})$ obey **half-integer condition**

$$h_j^{UV} + h_j^{IR} \in \frac{1}{2}\mathbb{Z}\,.$$

Definition

[2212.13851 (KK)]
[2412.08935 (KK)]

Theorem.

$$h_j^{UV} + h_j^{IR} \in \frac{1}{2}\mathbb{Z}$$

Proof.

- Let (h_i, h_j) be conformal dimensions of local operators.
- **Single-valued** correlation functions $\Rightarrow h_i - h_j \in \frac{1}{2}\mathbb{Z}$.
- **Chiral operators** should have $(h_i,0), (0,h_j)$ w/ $h_i, h_j \in \frac{1}{2}\mathbb{Z}$.
- As $c_j^{UV} c_j^{IR} \otimes 1 = c_j^{UV} c_j^{IR}$, the product object has chiral ops. w/

$$(h_j^{UV} + h_j^{IR},0), (0,h_j^{UV} + h_j^{IR}) . \quad \square$$

Definition

[2412.08935 (KK)]
[Gaiotto '12]

Where did we use RG domain wall?

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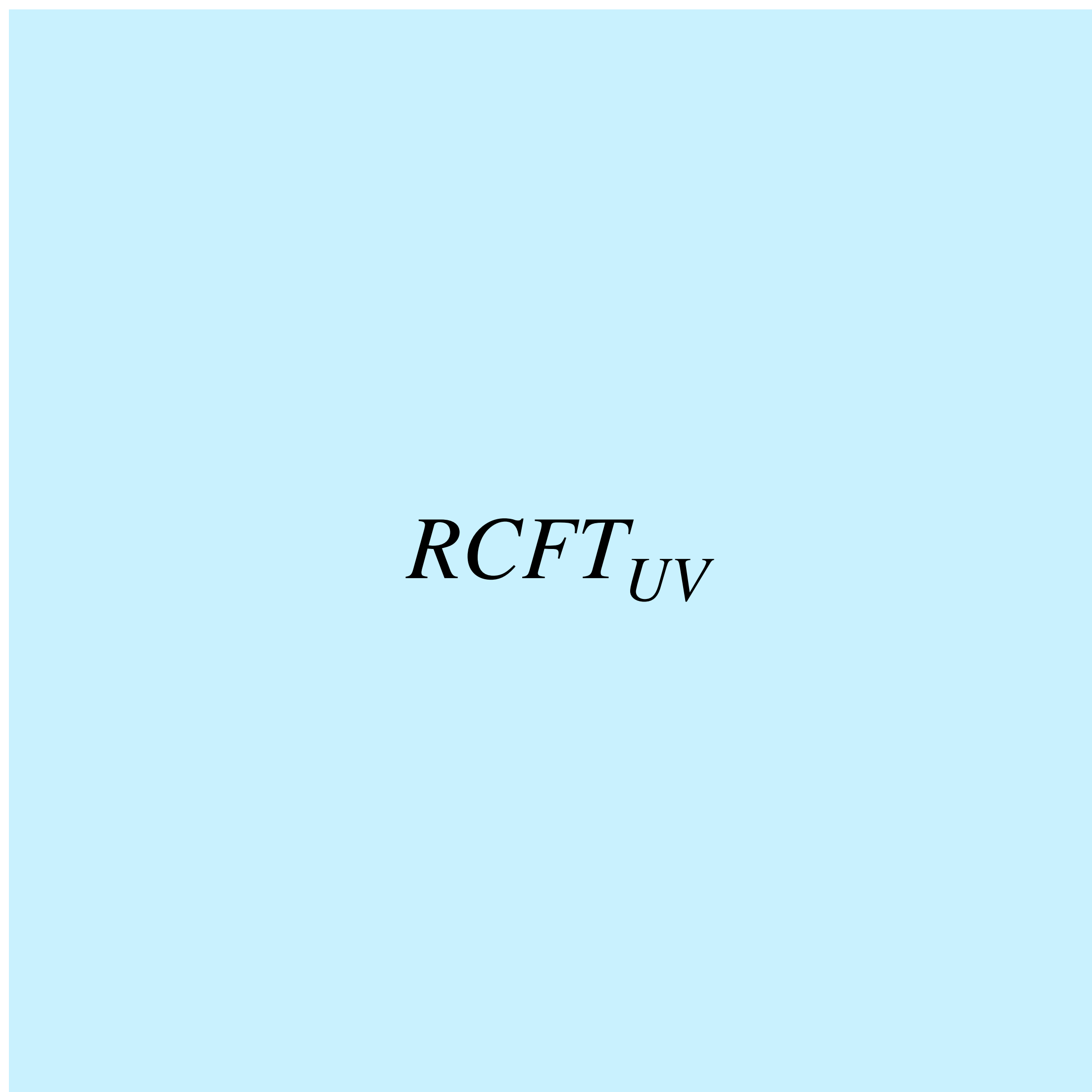

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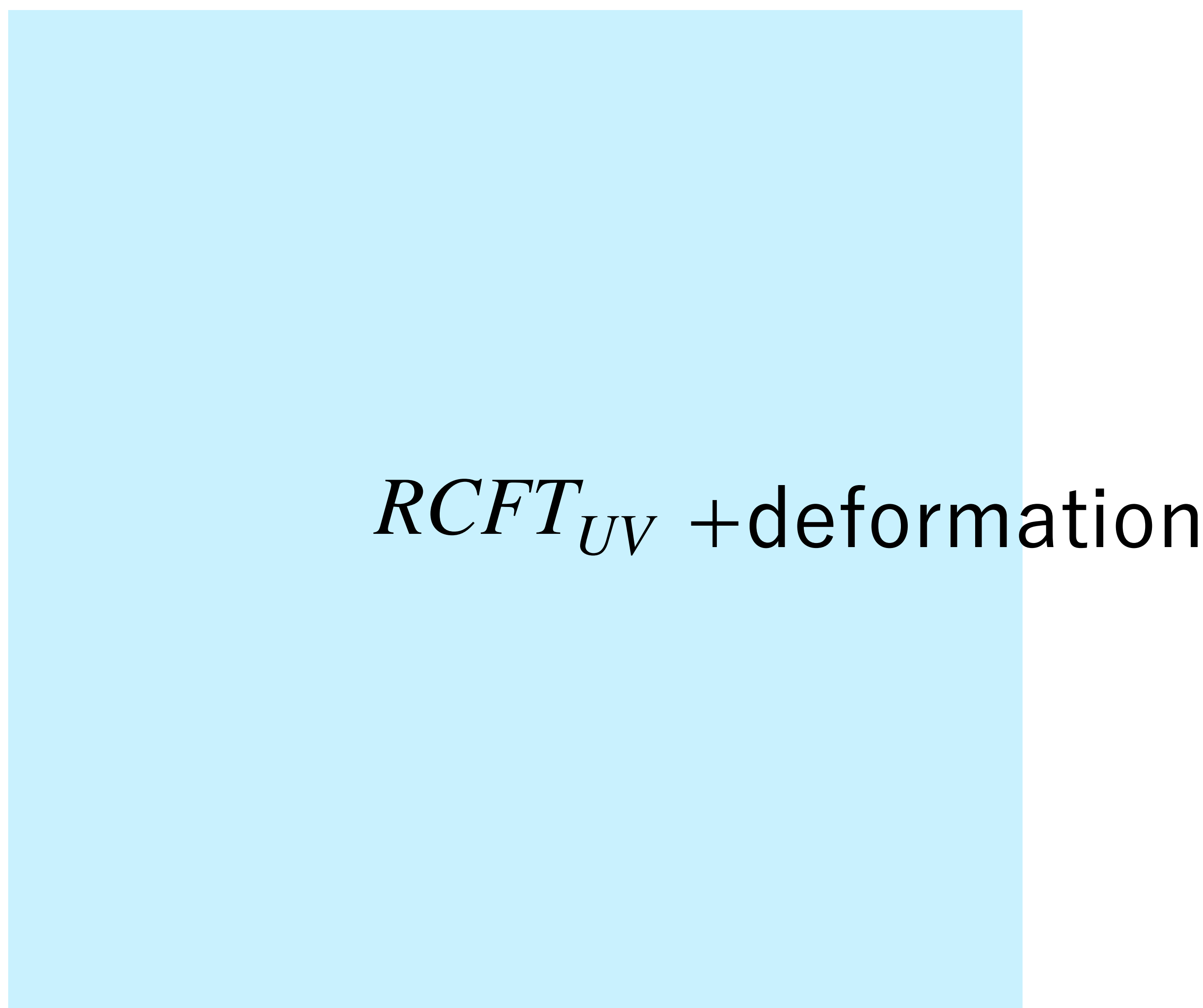

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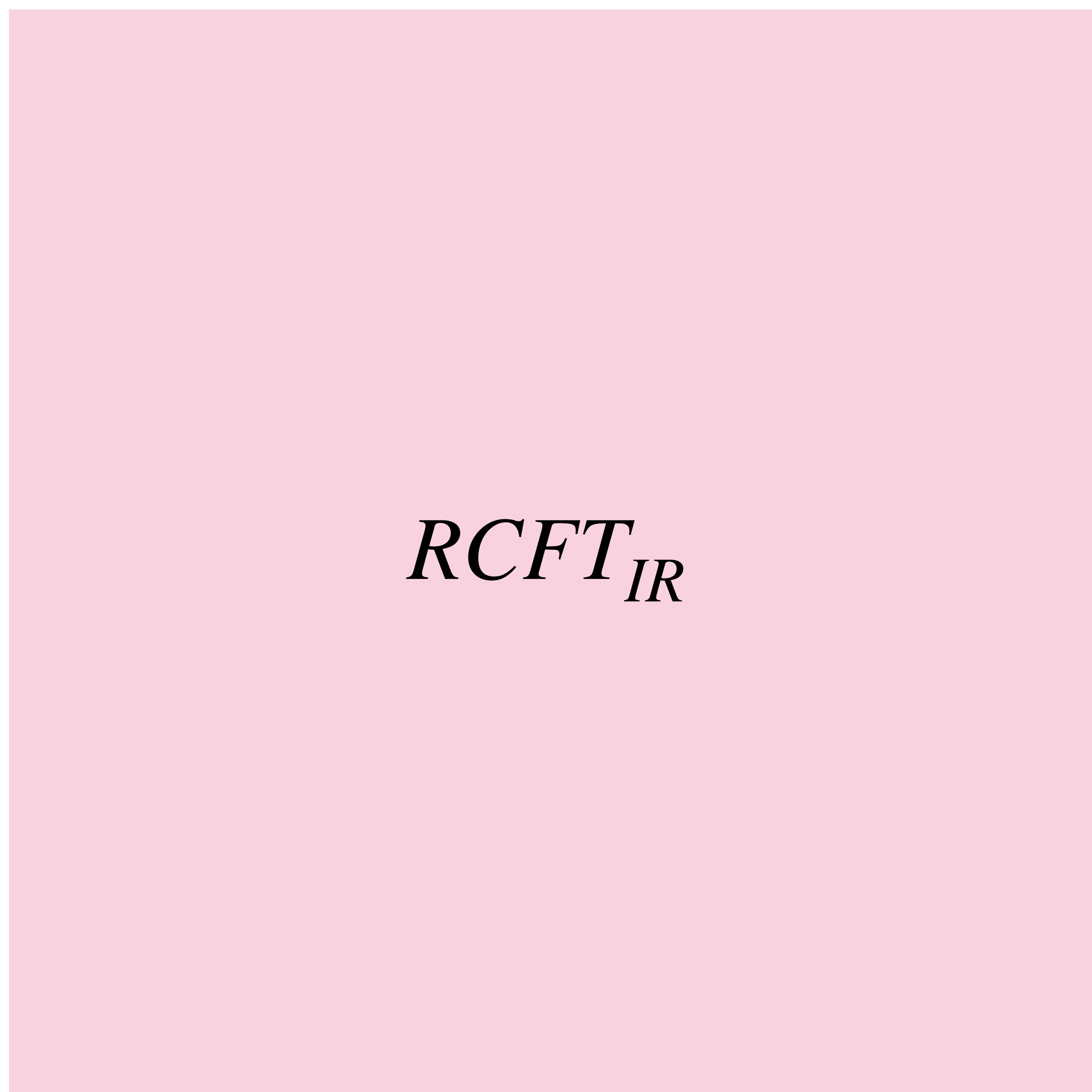

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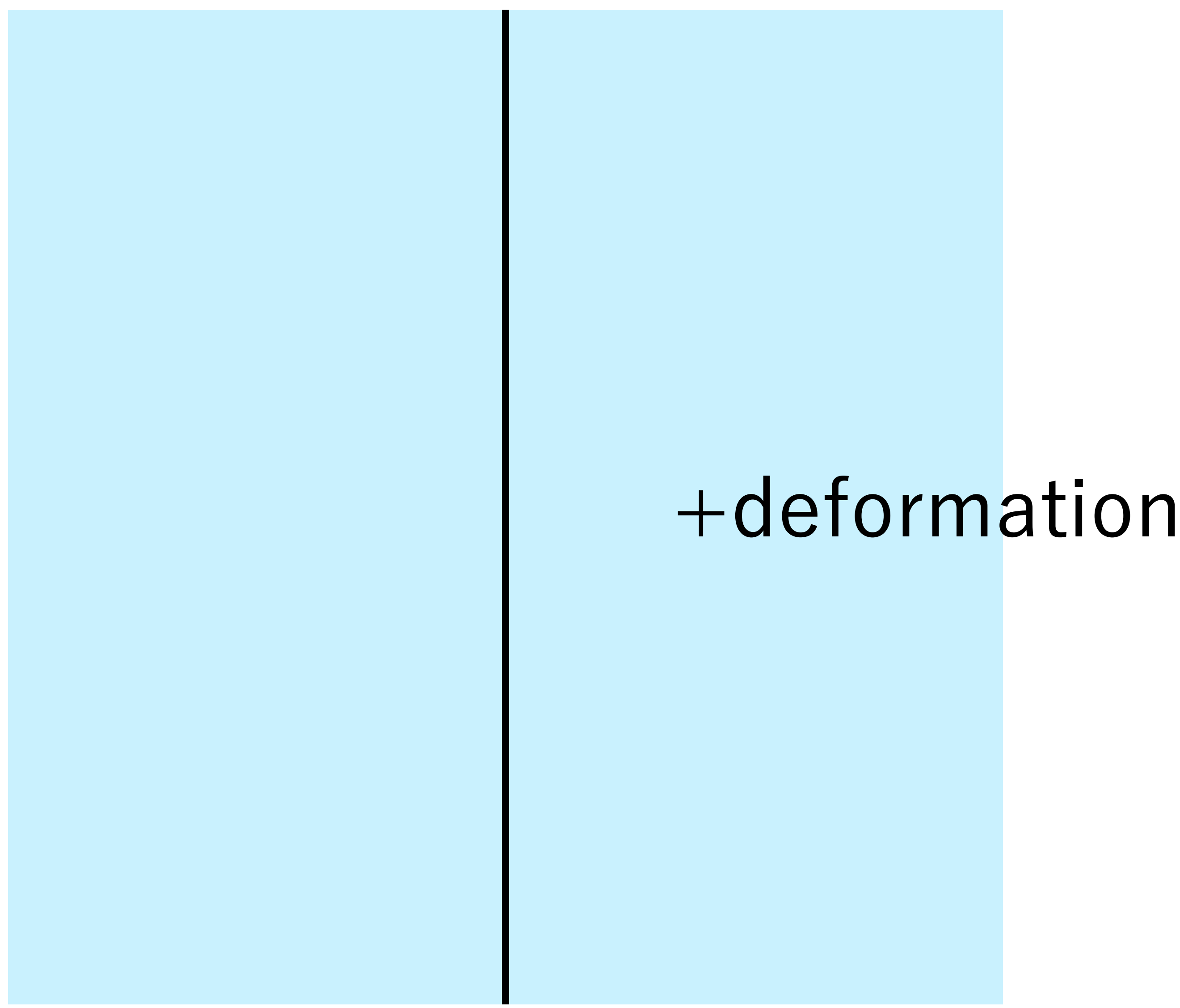

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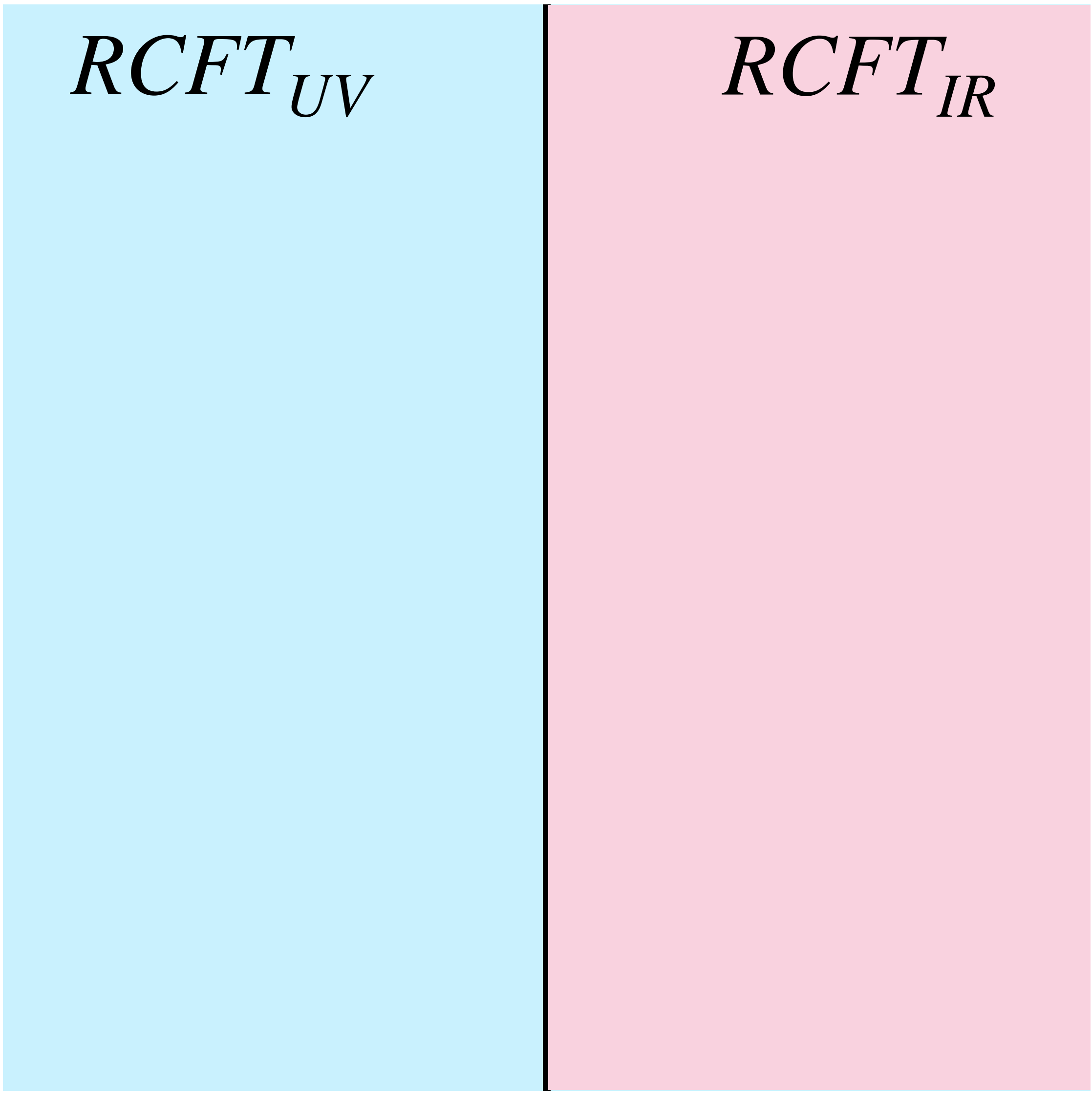

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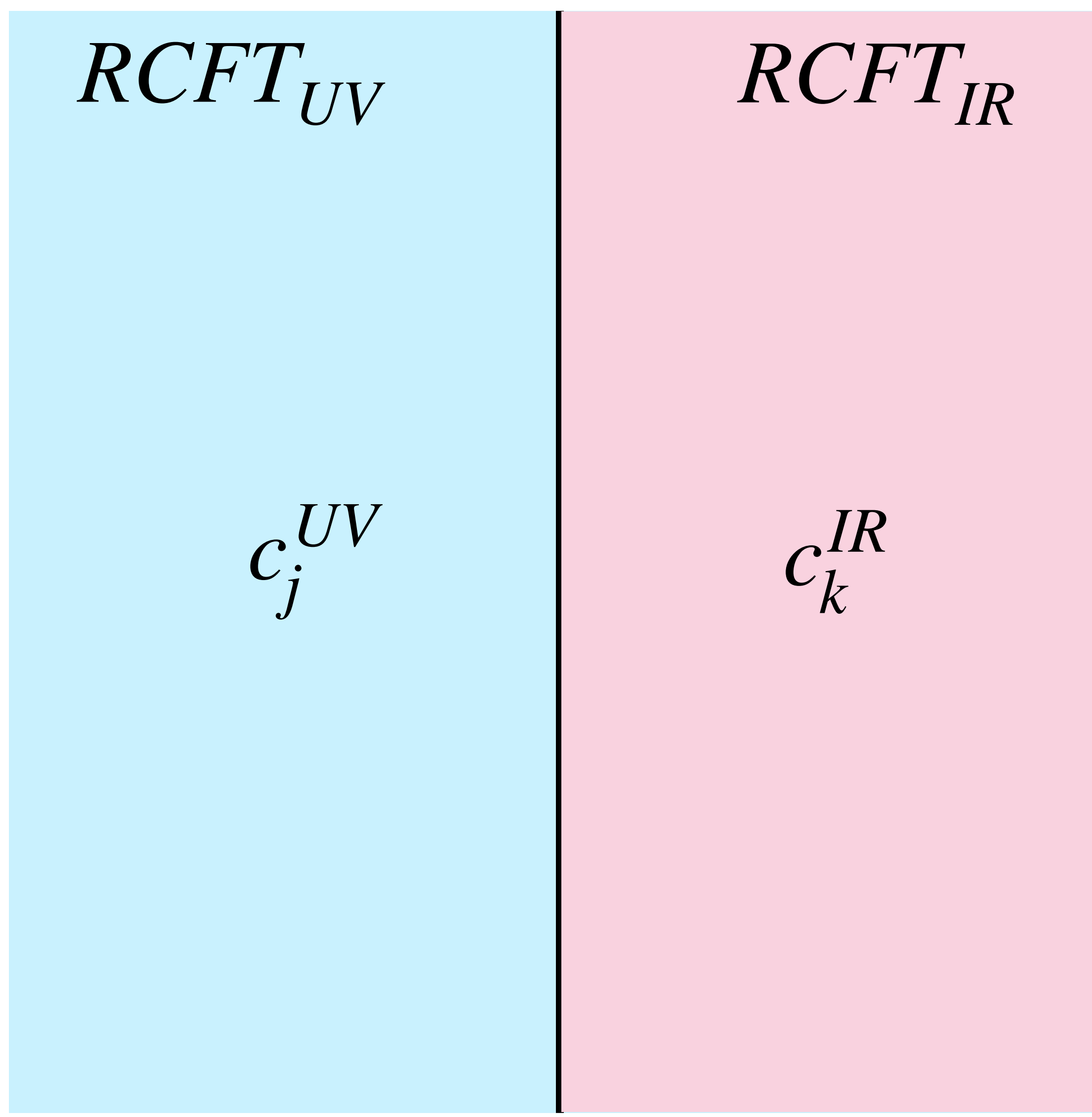

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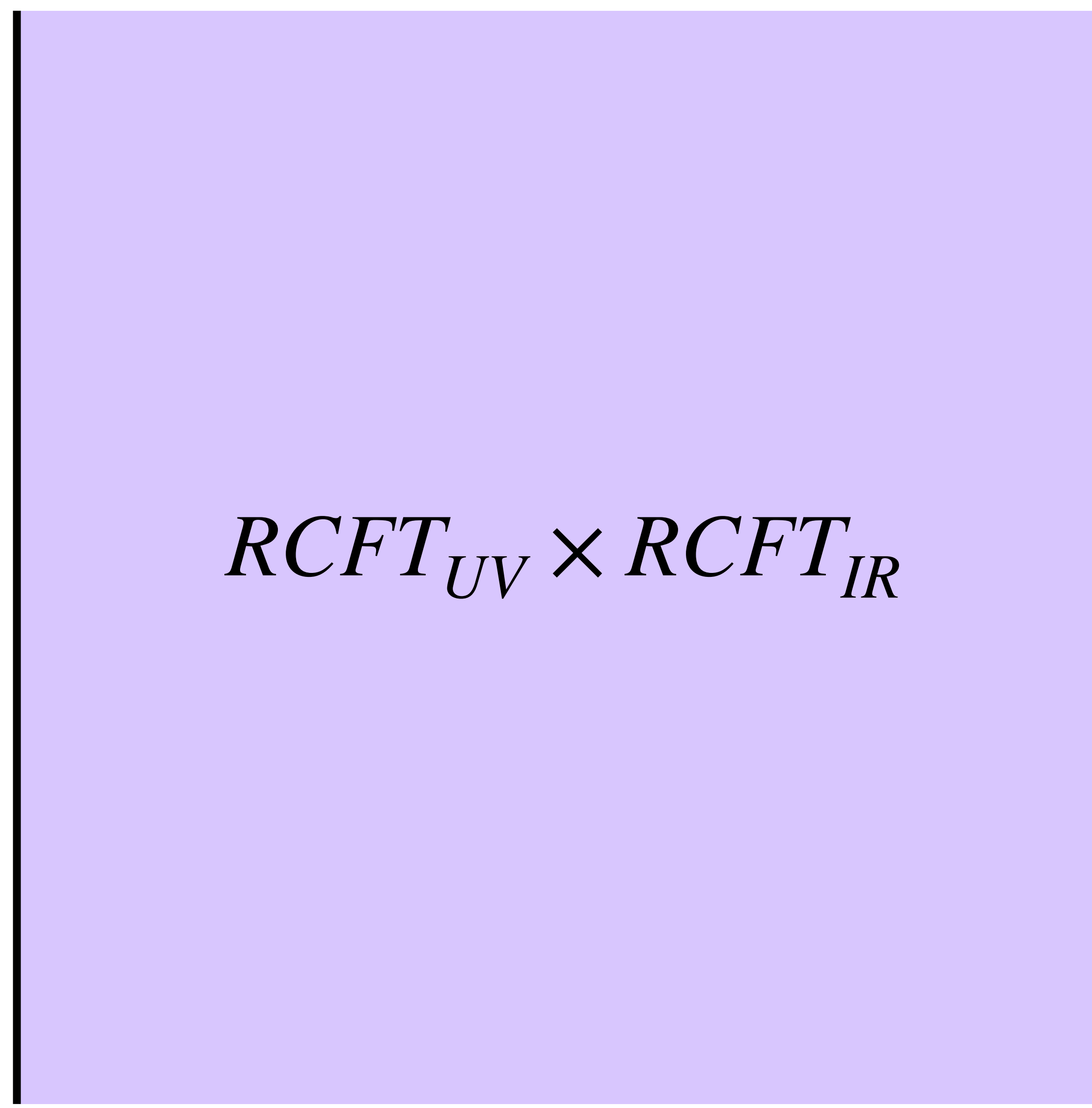

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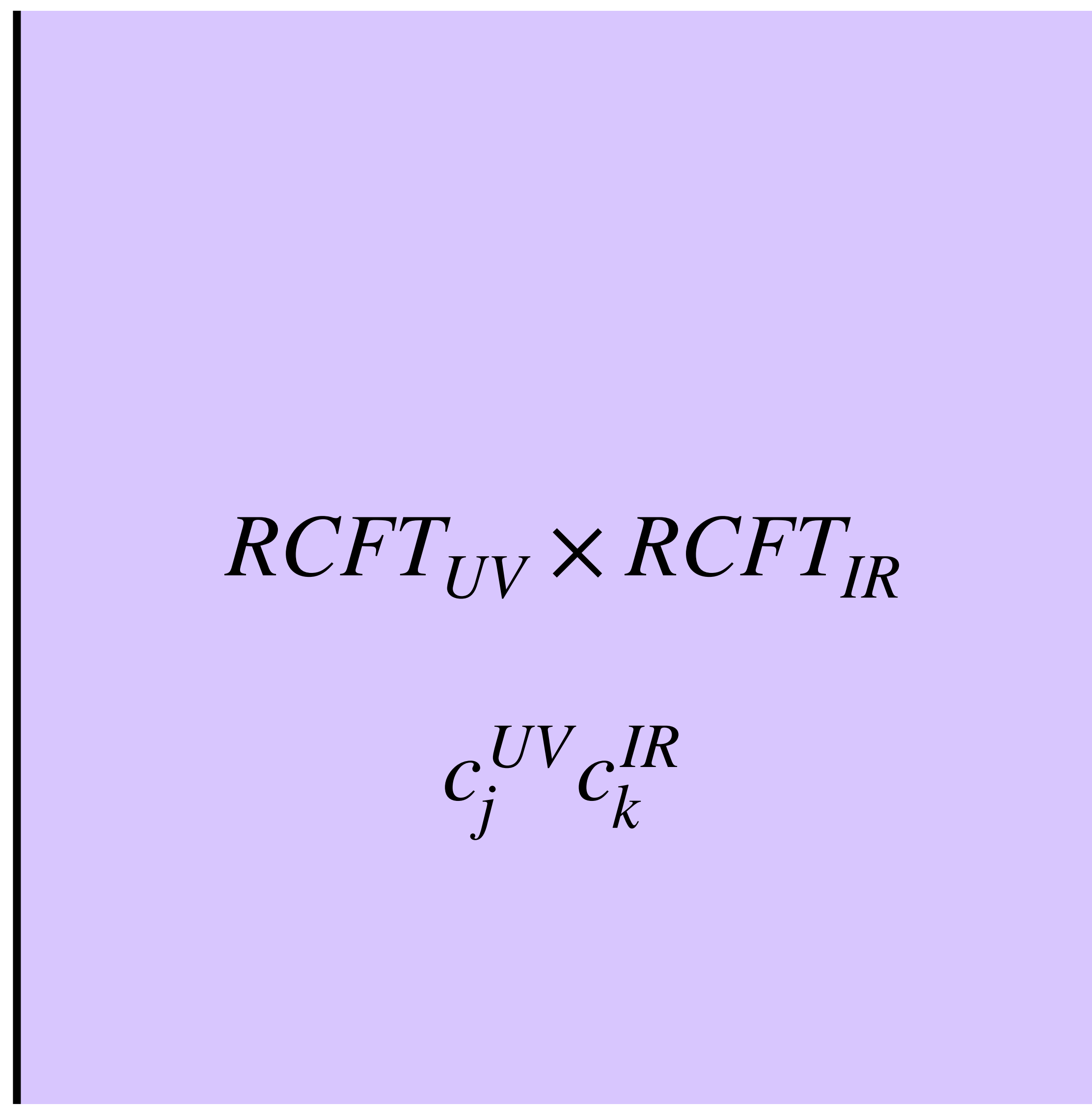

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Examples

[2412.08935 (KK)]

Example 1. Tricritical Ising $+\phi_{1,3} \to$ Critical Ising

The deformation preserves Ising MFC

$$C_{UV} = \{1, x, y\}.$$

They flow as

$$\begin{array}{rccc} \text{Tricritical Ising :} & 1_0 & x_{\frac{3}{2}} & y_{\frac{7}{16}} \\ & \downarrow & \downarrow & \downarrow \\ \text{Critical Ising :} & 1_0 & X_{\frac{1}{2}} & Y_{\frac{1}{16}} \end{array}$$

consistent w/ half-integer condition $(h_1, h_{xX}, h_{yY}) = (0, 2, \frac{1}{2})$.

Examples

[2412.08935 (KK)]

Example 1. Tricritical Ising $+\phi_{1,3} \to$ Critical Ising

``Normal subcat.'' $A = 1 \oplus xX$ gives ``quotient subcat.''

$$C_{UV} \boxtimes C_{IR}/A = \text{ToricCode}.$$

The MFC is ``simple.''

[2312.13353 (KK)]

Examples

[2412.08935 (KK)]

Example 2. Non-unitary $M(3,5) + \phi_{1,2} \to M(2,5)$

The deformation preserves Fibonacci MFC

$$C_{UV} = \{1, x\}.$$

They flow as

$$\begin{array}{cccc} M(3,5): & 1_0 & & x_{\frac{1}{5}} \\ & \downarrow & & \downarrow \\ M(2,5): & 1_0 & & X_{-\frac{1}{5}} \end{array}$$

consistent w/ half-integer condition $(h_1, h_{xX}) = (0,0)$.

Examples

[2412.08935 (KK)]

Example 2. Non-unitary $M(3,5) + \phi_{1,2} \to M(2,5)$

``Normal subcat.'' $A = 1 \oplus xX$ gives ``quotient subcat.''

$$C_{UV} \boxtimes C_{IR}/A = \{1\}.$$

The MFC is ``simple.''

Examples

[2412.08935 (KK)]

Example 3. Wess-Zumino-Witten $\hat{su}(3)_2 + \phi_{adj} \to \hat{su}(3)_1$

The deformation preserves $\mathbb{Z}_3$ MFC

$$C_{UV} = \{1, x, y\}.$$

They flow as

$$\begin{array}{cccc} \widehat{\mathfrak{su}}(3)_2: & 1_0 & x_{\frac{2}{3}} & y_{\frac{2}{3}} \\ & \downarrow & \downarrow & \downarrow \\ \widehat{\mathfrak{su}}(3)_1: & 1_0 & X_{\frac{1}{3}} & Y_{\frac{1}{3}} \end{array}$$

consistent w/ half-integer condition $(h_1, h_{xX}, h_{yY}) = (0,1,1)$.

Examples

[2412.08935 (KK)]

Example 3. Wess-Zumino-Witten $\hat{su}(3)_2 + \phi_{adj} \to \hat{su}(3)_1$

``Normal subcat.'' $A = 1 \oplus xX \oplus yY$ gives ``quotient subcat.''

$$C_{UV} \boxtimes C_{IR}/A = \{1\}.$$

The MFC is ``simple.''

Let's solve new RG flow
armed w/ the **new constraint**

Examples

[2412.08935 (KK)]

Example 4. E-type $(A_{10}, E_6) + \phi_{h=\frac{1}{11}} \to$ Critical Ising

The deformation preserves Ising MFC

$$C_{UV} = \{1, x, y\}.$$

As $F : C_{UV} \to C_{IR}$ is monoidal, we know $C_{IR} =$ Ising FC $\{1, X, Y\}$.

[Drinfeld-Gelaki-Nikshych-Ostrik '09]

They have $(h_X, h_Y) = (\frac{1}{2}, \frac{2n+1}{16})$ w/ $n \in \mathbb{Z}$.

Examples

[2412.08935 (KK)]

Example 4. E-type $(A_{10}, E_6) + \phi_{h=\frac{1}{11}} \to$ Critical Ising

Assume the flow admits RG domain wall à la Gaiotto. [Gaiotto '12]

Then, h_j^{IR} obeys half-integer condition.

h_X is automatic, but h_Y can only take $h_Y = \frac{1}{16}, \frac{9}{16}$.

There is no RCFT w/ $c < \frac{21}{22} = c_{(A_{10},E_6)}, h_Y = \frac{9}{16}$.

The only possibility is $h_Y = \frac{1}{16}$, critical Ising.

Examples

[2412.08935 (KK)]

Example 4. E-type $(A_{10}, E_6) + \phi_{h=\frac{1}{11}}$ → Critical Ising

They flow as

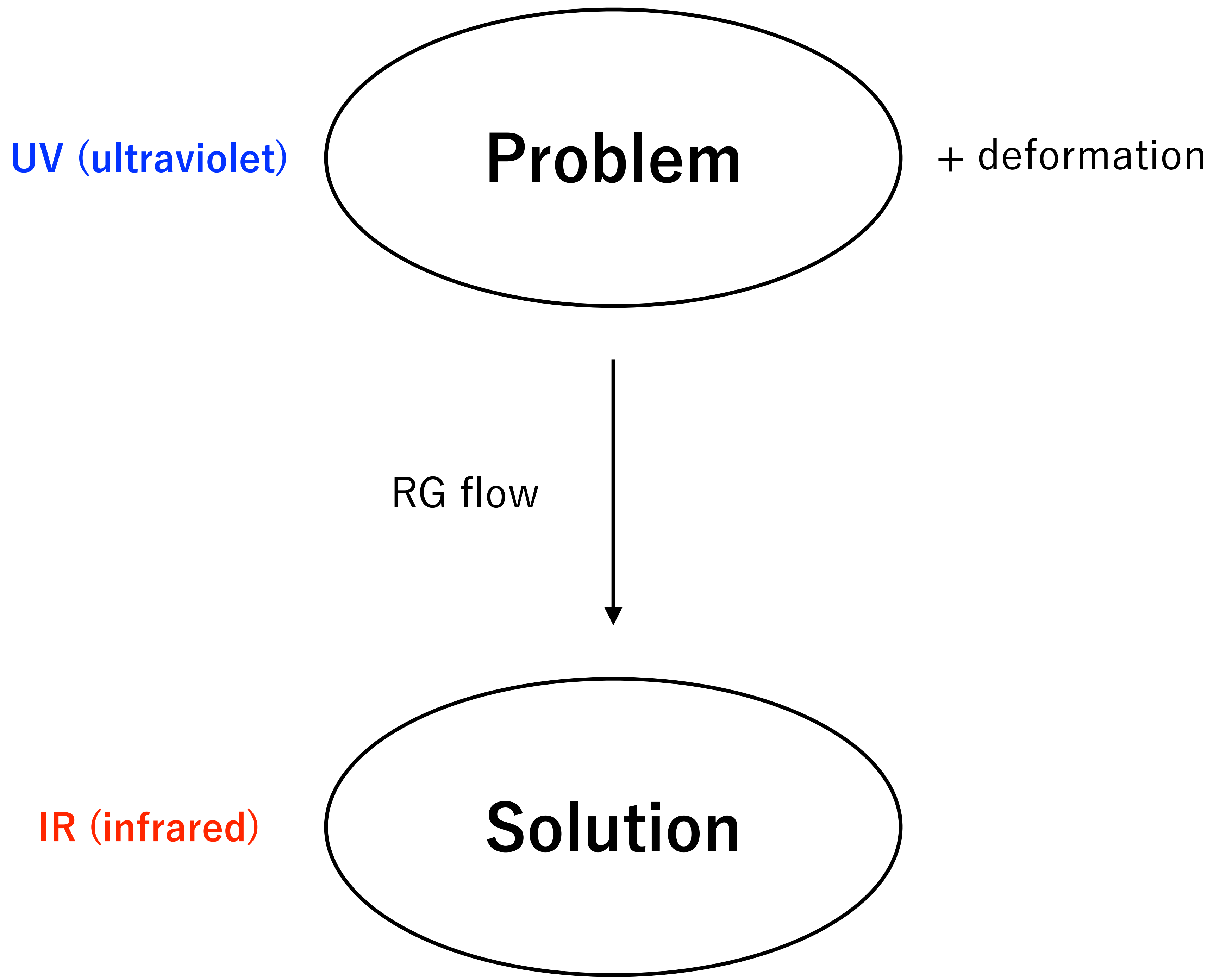
$$\begin{array}{rccc} (A_{10}, E_6): & 1_0 & x_{\frac{7}{2}} & y_{\frac{31}{16}} \\ & \downarrow & \downarrow & \downarrow \\ \text{Critical Ising}: & 1_0 & X_{\frac{1}{2}} & Y_{\frac{1}{16}} \end{array}$$

consistent w/ half-integer condition $(h_1, h_{xX}, h_{yY}) = (0,4,2)$.

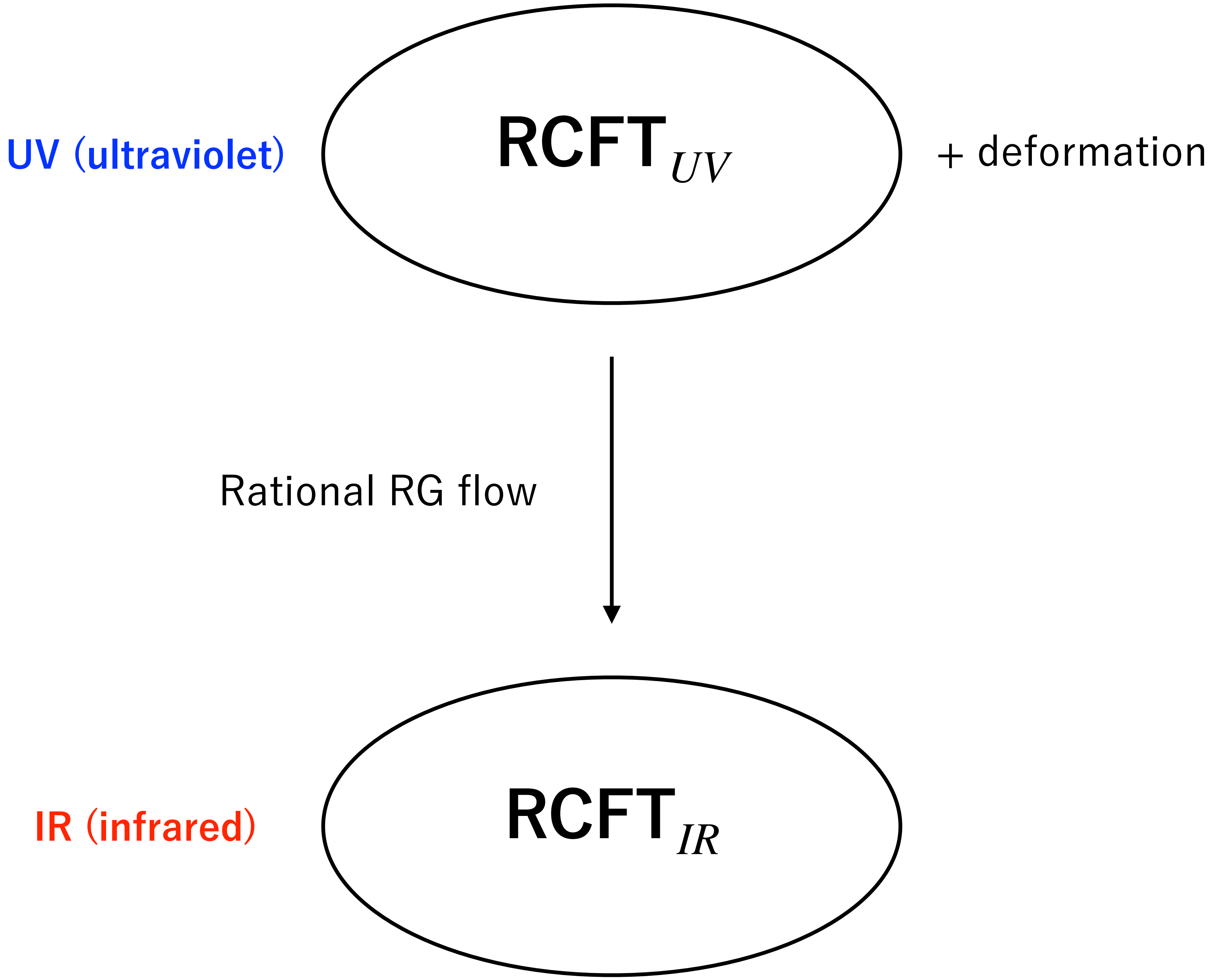
``Normal subcat.'' $A = 1 \oplus xX \oplus yY$ gives ``simple quotient''

$$C_{UV} \boxtimes C_{IR}/A = \{1\}.$$

Summary


UV (ultraviolet)
Problem
+ deformation
RG flow
IR (infrared)
Solution

Summary


UV (ultraviolet)
RCFT_{UV}
+ deformation
Rational RG flow
IR (infrared)
RCFT_{IR}

Summary

[2412.08935 (KK)]

- Discrete quantities can **jump** under RG flows.

- Gave **definition** $\{$Rational RG flow w/ MFC $C_{UV}\}$
  $\cong \{$Completely anisotropic representative of $[C_{UV} \boxtimes C_{IR}]\}$.

- Proved **half-integer condition** $h_j^{UV} + h_j^{IR} \in \frac{1}{2}\mathbb{Z}$.

[2212.13851 (KK)]

- Used the new constraint to **solve** $(A_{10}, E_6) \to$ Critical Ising.