Topological symmetries and their gaugings in 2d CFT and 3d TFT

Ingo Runkel (Hamburg Univ.)

Outline

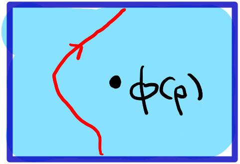
Part I Topological line defects in 2d QFT & 3d TFT

Part II Applications

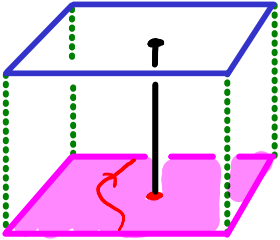
- Gauging topol. symm. in 2d QFT
- Gauging as an equiv. rel.
- Non-topol. defects & integrable flows

Part I

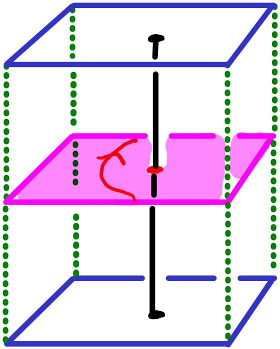
2d QFT with topological line defects



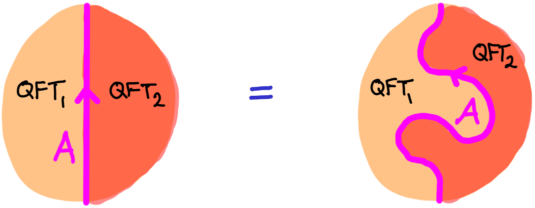
Symmetry topol. field theory ("SymTFT")



Chiral TFT



Topological defects

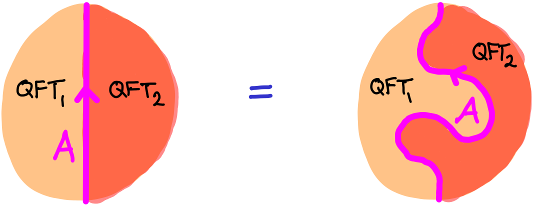


For now: $QFT_1 = QFT_2$

The symmetries of a quantum mech. system form a group.

The topol. line defects of a 2d QFT form a $\boxed{}$.

Topological defects



For now: $QFT_1 = QFT_2$

The symmetries of a quantum mech. system form a group.

The topol. line defects of a 2d QFT form a $\boxed{}$.

pivotal monoidal category

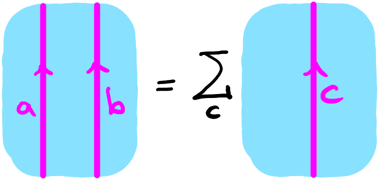
Fuchs, Schweigert, IR '02
Fröhlich, Fuchs, Schweigert, IR '09
Davydov, Kong, IR '11
Thorngren, Wang '19

Fusion categories

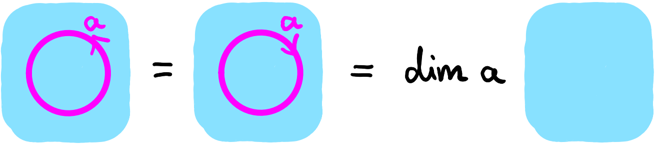
Assume :

- finite list of elementary top. def. $\{1, a_2, \dots, a_n\}$
- closed under fusion

  $$a \otimes b \simeq \bigoplus c$$



- quantum dimensions



spherical fusion category

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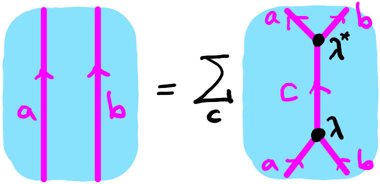
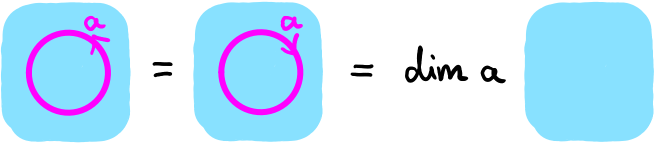


assume 0 or 1 dim. coupling spaces

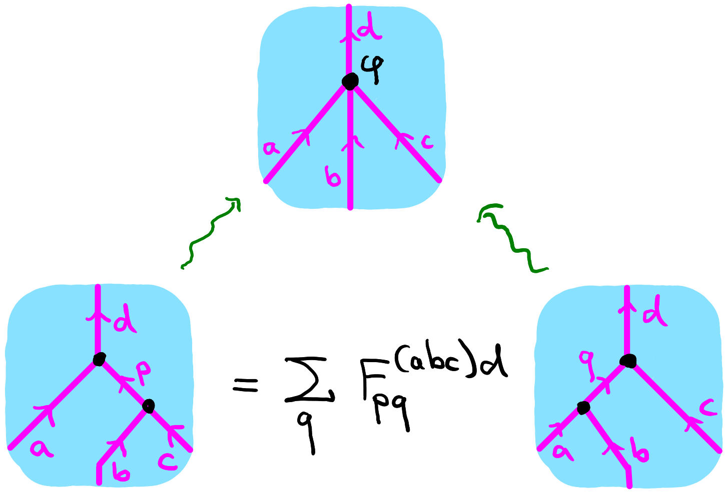
- quantum dimensions



spherical fusion category

... fusion categories

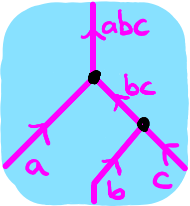
F-matrices : Change of basis in $a \otimes b \otimes c \to d$



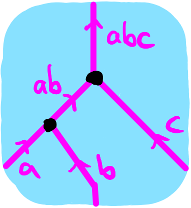
... fusion categories

Group symmetry : $a, b, \ldots \in G$ finite group

$a \otimes b \simeq c$ (no sum)



$= \omega(a,b,c) \cdot$



$[\omega] \in H^3(G, \mathbb{C}^\times)$

obstruction to setting $\omega \equiv 1$

$\hookrightarrow$ obstruction to gauging G

... fusion categories

E.g. $G = \mathbb{Z}_2$ $\quad H^3(\mathbb{Z}_2, \mathbb{C}^\times) \simeq \mathbb{Z}_2$

$$\left.\begin{array}{l} H^3(\mathbb{Z}_2, \mathbb{C}^\times) \simeq \mathbb{Z}_2 \\ \{d : \mathbb{Z}_2 \to \mathbb{C}^\times\} \simeq \mathbb{Z}_2 \end{array}\right\} \text{4 choices}$$

Write multiplicatively: $\mathbb{Z}_2 \simeq \{\pm 1\}$

	$\omega(\underline{1}, \underline{1}, \underline{1}) = 1$	$\omega(\underline{1}, \underline{1}, \underline{1}) = -1$
$d(\underline{1}) = 1$	$\mathrm{Vect}_{\mathbb{Z}_2}$	$\mathrm{Vect}^{\omega}_{\mathbb{Z}_2}$
$d(\underline{1}) = -1$	SVect	SVect^{ω}

... fusion categories

E.g. $G = \mathbb{Z}_2$ $\quad H^3(\mathbb{Z}_2, \mathbb{C}^\times) \simeq \mathbb{Z}_2$

$\{d : \mathbb{Z}_2 \to \mathbb{C}^\times\} \simeq \mathbb{Z}_2$ $\Big\}$ 4 choices

Write multiplicatively : $\mathbb{Z}_2 \simeq \{\pm 1\}$

	$\mathbb{Z}_2$ can be gauged $\omega(\underline{1}, \underline{1}, \underline{1}) = 1$	$\mathbb{Z}_2$ cannot be gauged $\omega(\underline{1}, \underline{1}, \underline{1}) = -1$
$d(\underline{1}) = 1$	$\mathrm{Vect}_{\mathbb{Z}_2}$ gauging well-defined on oriented surfaces	$\mathrm{Vect}^\omega_{\mathbb{Z}_2}$
$d(\underline{1}) = -1$	SVect gauging needs a spin structure	SVect^ω

... fusion categories

E.g.: simplest non-group case: Fib

- simple obj $\mathbb{1}$, φ
- tensor $\varphi \otimes \varphi \simeq \mathbb{1} \oplus \varphi$
- F-matrix : 2 choices
- dim's : 1 choice

Spherical fusion cat. classified for:

$\#(\text{simple obj}) =$

... fusion categories

E.g.: simplest non-group case: Fib

- simple obj $\mathbb{1}, \varphi$
- tensor $\varphi \otimes \varphi \simeq \mathbb{1} \oplus \varphi$
- F-matrix : 2 choices
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Spherical fusion cat. classified for: Ostrik '02, '13

$\#(\text{simple obj}) = 1, 2, 3$

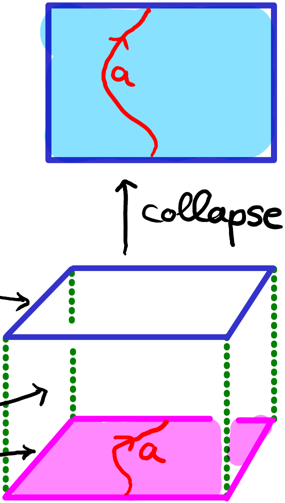
Symmetry topological field theory

Gaiotto, Kulp '20
Freed, Moore, Teleman '22
...

Q 2dQFT with spherical fusion cat. $\mathcal{F}$ of topol. line defects

Get

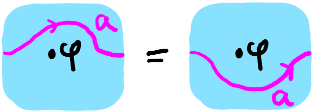
- $Z_{\mathcal{F}}$ 3d state sum TFT
- non-top. bnd. cond. $\tilde{Q}$
- top. bnd. cond. $\tilde{\mathcal{F}}$

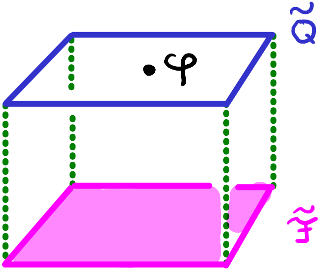


... SymTFT

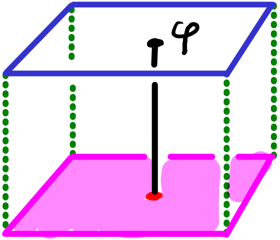
Bulk fields separate

• commute with all $a \in \mathcal{F}$





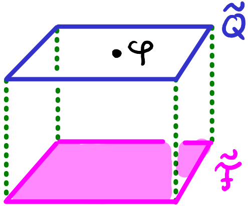
• do not commute with all $a \in \mathcal{F}$



Chiral TFT

Special to 2d CFT : E.g. Ising CFT

choice of $\mathcal{F}$	bulk field commutes? $\mathbb{1}$ $(0,0)$	ε $(\frac{1}{2},\frac{1}{2})$	σ $(\frac{1}{16},\frac{1}{16})$
id	✓	✓	✓
$\mathbb{Z}_2$	✓	✓	✗
$\mathbb{Z}_2 \cup \{KW\}$	✓	✗	✗



only (hol) $\otimes$ (antihol) lives on $\tilde{Q}$.

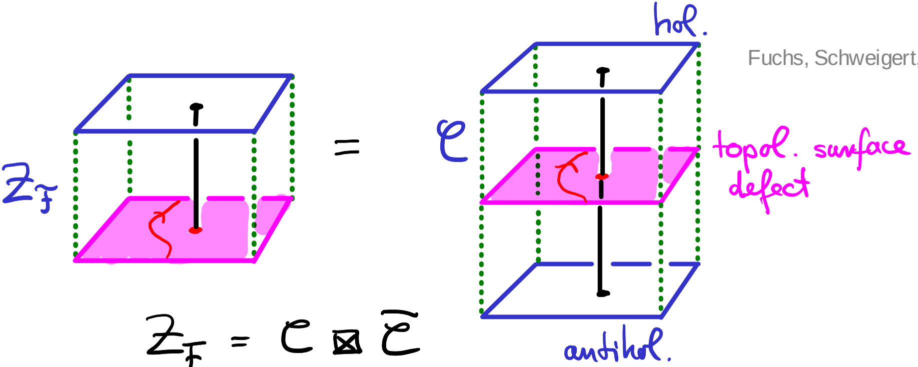
... chiral TFT

In rational CFT :

1) If only (hol) $\otimes$ (antihol) lives on $\tilde{Q}$, can "unfold" SymTFT $\rightsquigarrow$ chiral TFT

2) There is a choice of $\mathcal{F}$ s.th. 1) applies

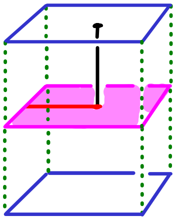
Fuchs, Schweigert, IR '02, ...

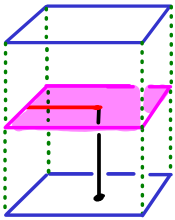


$$Z_{\mathcal{F}} = \mathcal{C} \boxtimes \overline{\mathcal{C}}$$

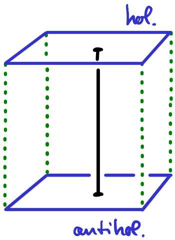
... chiral TFT

Nice :





- geometric separation into hol. / antihol.



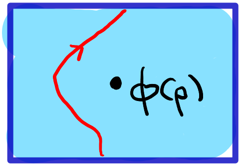
- very easy for identity surf. def.

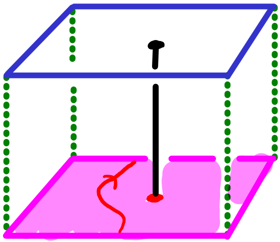
Part I summary:

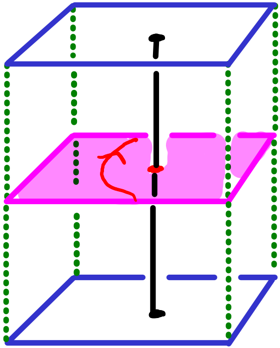
2d RCFT with topological line defects

Symmetry topol. field theory ("SymTFT")

Chiral TFT





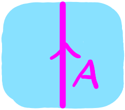


Now: Part II – Applications

Gauging topological symmetries in 2d
aka generalised orbifolds

Fröhlich, Fuchs, Schweigert, IR '09
Carqueville, IR '12

$\mathcal{Q}$: 2d QFT

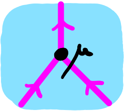


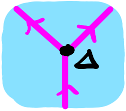
A : topol. def. in $\mathcal{Q}$
(not nec. elementary)

Want to define gauged theory $\mathcal{Q}/A$.

Also need :

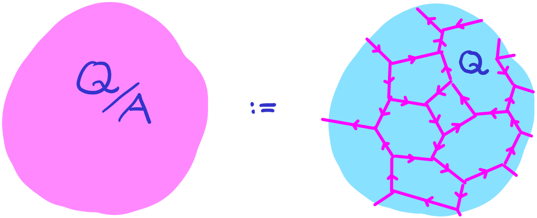
μ, Δ : topol. junctions



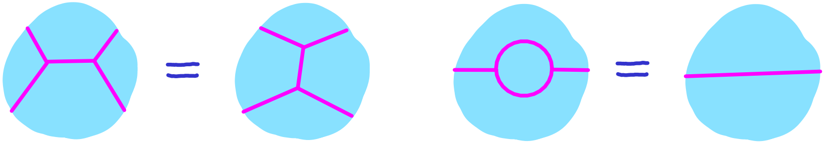


... gauging topol. sym.

Idea : Q/A := " Q with A-network "



Need : Invariance under change of A-network



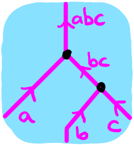
(for all orientations)

$\rightsquigarrow$ A is symmetric Δ-separable Frobenius alg. (assume unit)

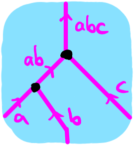
... gauging topol. sym.

Example: Gauge a symmetry group G

$$A = \bigoplus_{a \in G} a$$



$= \omega(a,b,c) \cdot$



$[\omega] \in H^3(G, \mathbb{C}^\times)$

Δ-sep. Frob. alg exists on A <u>if and only if</u>

$[\omega] = 1$ (obstruction vanishes)

$$\left\{ \begin{array}{l} \Delta\text{-sep. Frob. alg on } A \\ \text{up to isom.} \end{array} \right\} \simeq H^2(G, \mathbb{C}^\times)$$

... gauging topol. sym.

Fröhlich, Fuchs, Schweigert, IR '09

Thm.:

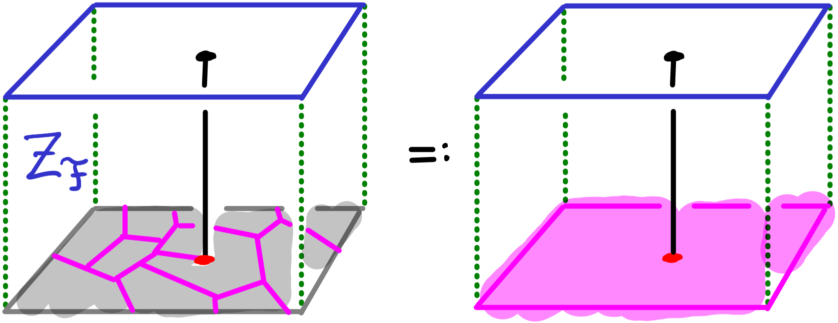
Any two rational 2d conformal field theories containing the chiral symmetry $V \otimes V$
(with unique vacuum and non-deg. 2pt correlators)
are obtained from each other by gauging top. sym.

implies same central charge

⚠ Wrong for just group symmetries!

... gauging topol. sym.

SymTFT picture:



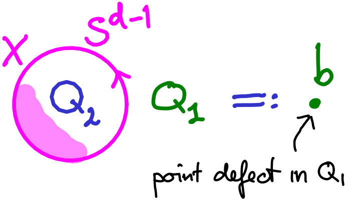
Any two top. bnd. of $Z_{\mathcal{F}}$ are related to each other by gauging.

Invertible bubbles and gauging

Carqueville, IR '12

Given: topol. defect separating d-dim QFTs Q_1 and Q_2

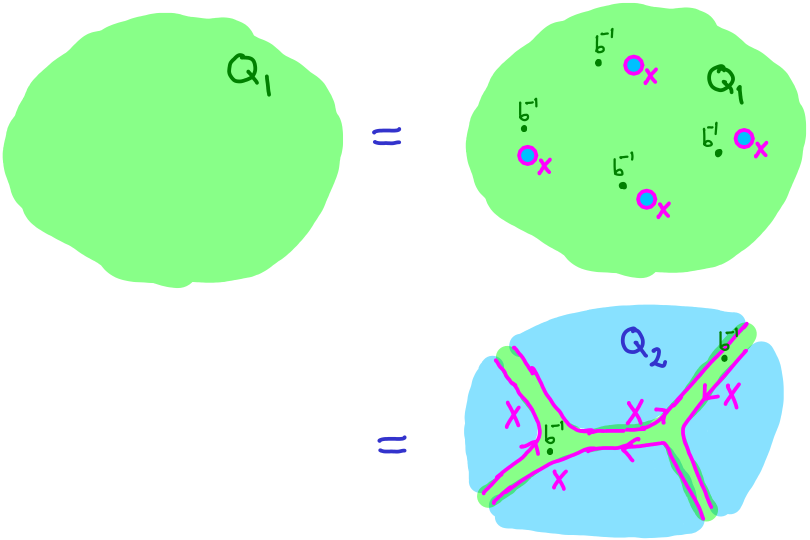
such that



is invertible as a point defect in Q_1 and vice versa.

Then: Q_1 is obtained from Q_2 by gauging a top. sym. and vice versa.

... invertible bubbles



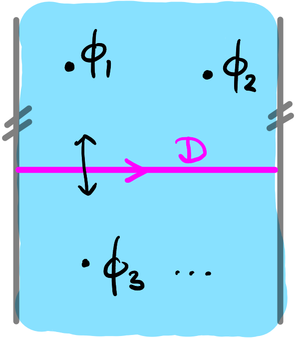
Get equivalence relation on d-dim. QFTs.

Perturbed defects and integrability

Ambrosino, Watts, IR: in prep.

$\mathcal{J}(Q)$: Rigidly translation invariant defects in 2d QFT Q

$$[H_Q, D] = 0$$



$\Rightarrow$ defect fusion is well-defined

$\mathcal{J}(Q)$ is a monoidal category

(morphisms : topological junctions)

… pert. def. & integrability

$\mathcal{J}(Q)$ extends monoidal cat. $\mathcal{J}_{top}(Q)$ of topol. defects

$$\mathcal{J}_{top}(Q) \subset \mathcal{J}(Q) \quad \text{(full monoidal subcategory)}$$

<u>Idea</u>: Consider $\mathcal{J}(Q)$, not just $\mathcal{J}_{top}(Q)$ to get information about Q and flows $Q \to Q'$

... pert. def. & integrability

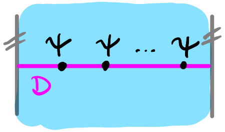
Bazhanov, Lukyanov, Zamolodchikov '96
IR '07

Apply to 2d CFT $\mathcal{C}$

D : topol. defect $\quad \psi$: holom. field on D

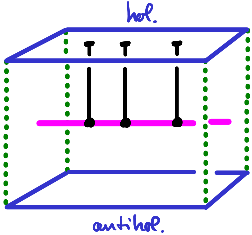
$\qquad\qquad\qquad\qquad \bar{\psi}$: antihol. — " —

Perturbed defects $(\lambda \in \mathbb{C})$



$$D(\lambda\psi) = \exp\left(\lambda\int_0^L \psi\right) \text{ on } D$$

$$D(\lambda\bar{\psi}) = \exp\left(\lambda\int_0^L \bar{\psi}\right) \text{ on } D$$



are rigidly transl. inv. since

$$\frac{\partial}{\partial t}\psi(s+it) = -i\frac{\partial}{\partial s}\psi(s+it)$$

... pert. def. & integrability

Manolopoulos, IR '09
Bücher, IR '12

Get monoidal subcat. $\underline{\mathcal{J}_{pert}(G)} \subset \mathcal{J}(G)$

- not semisimple
- abelian (kernels, quotients, ...)
- continuously many simple obj
- rigid (has duals), not spherical
- $\otimes$ of simples generically simple

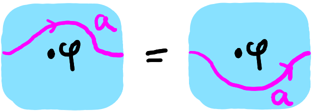
... pert. def. & integrability

Bazhanov, Lukyanov, Zamolodchikov '99
IR '10, Bücher, IR '12
Ambrosino, Watts, IR: in prep.

Perturb CFT by bulk field φ $\rightsquigarrow$ $C[\mu\varphi]$

What can we say about $\mathcal{J}(\mu)$?

$\mathcal{J}_{top}(\mu)$: topol. def a s.th.



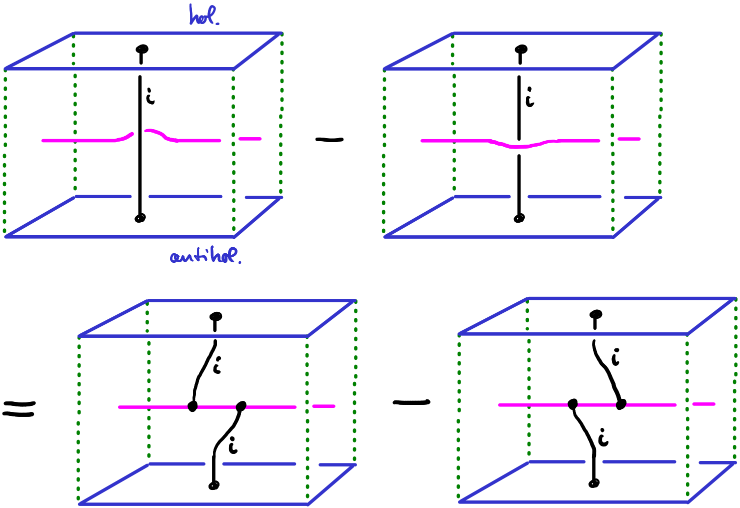
$\mathcal{J}_{pert}(\mu)$: iff commutation condition (→ next slide) holds

$\lambda \in \mathbb{C}^\times$

$$\Big[\underbrace{H_{CFT} + 2i\mu\int_0^L \varphi(s)\,ds}_{\text{perturbed Hamiltonian}} \;,\; \underbrace{D\big(\lambda\psi + \tfrac{\mu}{\lambda}\bar{\psi}\big)}_{D \text{ perturbed by } \psi \text{ and } \bar{\psi}}\Big] = 0$$

... pert. def. & integrability

The commutation condition (in chiral TFT with trivial surf. def.)



... pert. def. & integrability

If commutation condition holds:

$D(\lambda\Psi + \frac{\mu}{\lambda}\bar{\Psi})$ is one-param. family in $\mathcal{J}(\mu)$

$\rightsquigarrow$ family of non-local conserved charges in perturbed CFT $C(\mu\varphi)$.

... pert. def. & integrability

$$D\left(\lambda\Psi + \tfrac{\mu}{\lambda}\bar{\Psi}\right) \in \mathcal{J}_{pert}(\mu) \subset \mathcal{J}(\mu)$$

→ monoidal cat. $\mathcal{J}_{pert}(\mu)$ can be computed in UV CFT $\mathcal{C}$

→ In examples: D's commute for different $\lambda \in \mathbb{C}^\times$

$$\boxed{\text{sign of integrability}}$$

... pert. def. & integrability

Ambrosino, Watts, IR: in prep.

Example: A-type Virasoro minimal models $M(p,q)$

Perturbing bulk field $\varphi_{(r,s)}$

$(1,2)$, $(1,3)$, $(1,5)$ $\Big\}$

- solutions to comm. cond. exist
- $D(\lambda\psi + \frac{\mu}{\lambda}\bar{\psi})$ mutually commute

$(1,4)$, $(1,6)$, $(1,7)$, ... $\Big\}$

- solutions sporadically exist
- pert. defect often would need reg.

e.g. in $M(p,q)$, $q = 9, 10, 18$

Outlook

Further applications in 2d CFT

- functional relations & integral eqn for D's
- spin from oriented QFTs by gauging
- lattice models with fusion cat. topol. sym. from 2d CFT
- gauging non-inv. surface defects in 3d TFT
- non-semisimple ("logarithmic") 2d CFT and chiral / SymTFT

⋮