Boundary Scattering and Non-invertible Symmetries in (1+1) Dimensions

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Based on the collaboration with Satoshi Yamaguchi (Osaka U.)

[To appear]

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Outline

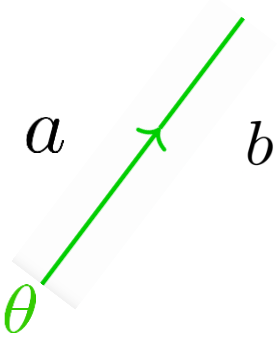
1. Introduction
2. Non-invertible symmetries and kinks
3. Boundary scattering
4. Modified boundary crossing relation
5. Summary and prospects

1. Introduction

- Scattering amplitude is a key physical quantity in QFTs .
- In general, it is quite hard to derive the exact S-matrix .
  non-perturbatively
- However, if a QFT in (1+1)d has the integrability, we can determine the S-matrix by the axiomatic way .

- Our interest in this talk is the kink scattering .
- Kink : massive excitation interpolating two adjacent vacua $|a\rangle$ and $|b\rangle$

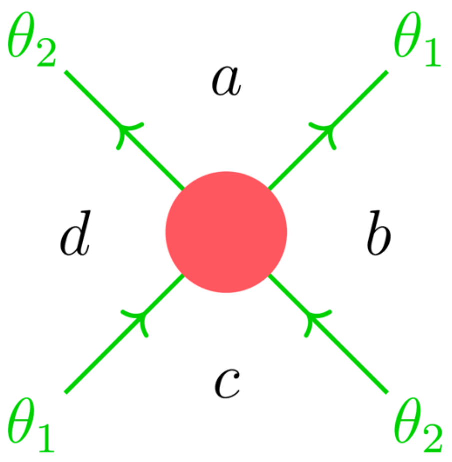
$$|K_{ab}(\theta)\rangle \quad =$$



(θ : rapidity)

- We can introduce the S-matrix for two-body kink scattering :

$$S^{ab}_{dc}(\theta) \quad = $$



- Integrability allows us to factorize all scattering amplitudes into the product of two-body scattering defined above.

- There are three consistency conditions for the S-matrix.

  (unitarity, Yang-Baxter equation, crossing relation)

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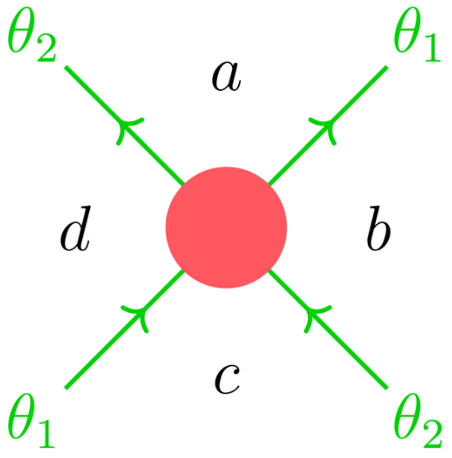

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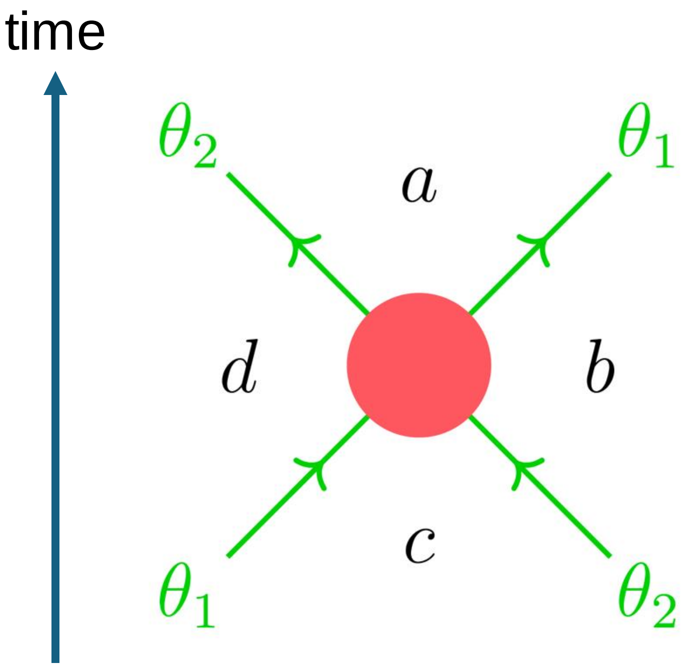
■ Crossing relation : $S_{dc}^{ab}(\theta) = S_{ad}^{bc}(\mathrm{i}\pi - \theta)$



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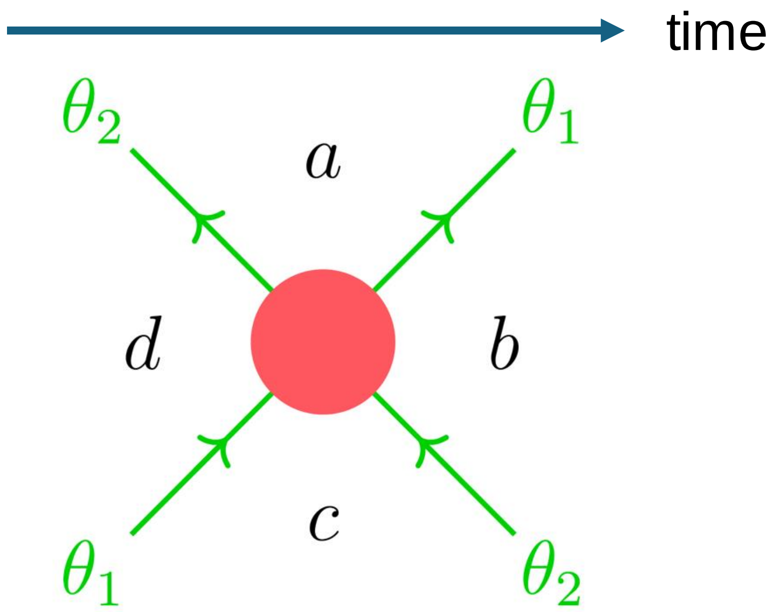

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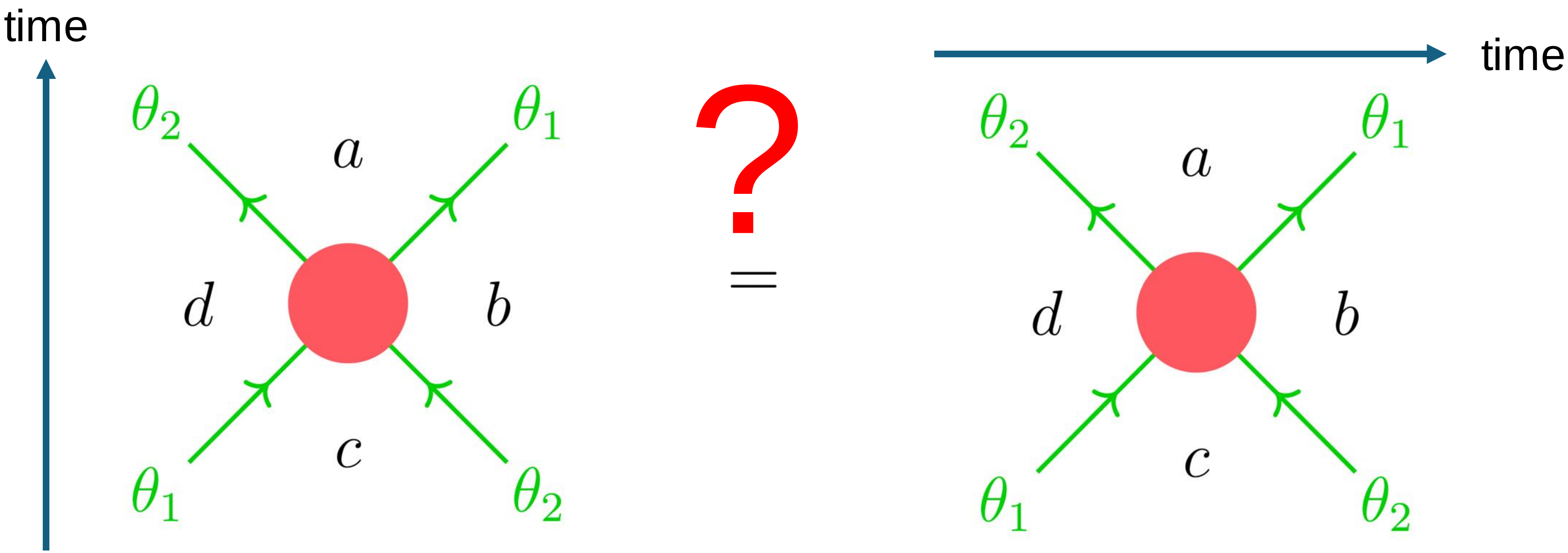


- When non-invertible symmetries are spontaneously broken in the IR gapped phase, the crossing relation must be modified as

$$S_{cd}^{ab}(\theta) = \sqrt{\frac{\mathrm{d}_a \mathrm{d}_c}{\mathrm{d}_b \mathrm{d}_d}} S_{ad}^{bc}(\mathrm{i}\pi - \theta)$$

[Copetti, Cordova, Komatsu, '24]

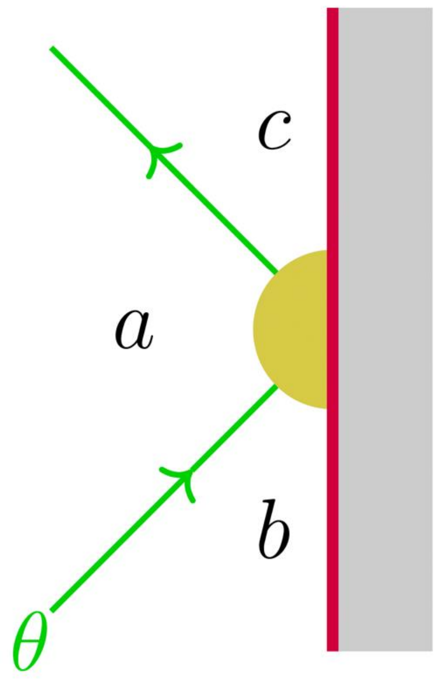
d_a : quantum dimension of topological defect

If we have a boundary in (1+1)d spacetime,

we can also define the boundary S-matrix .

[Ghoshal, Zamolodchikov, '94]

$$R^a_{bc}(\theta) \quad = $$



- Ghoshal and Zamolodchikov proposed the boundary crossing relation:

$$R_{ca}^{b}\left(\frac{\mathrm{i}\pi}{2}-\theta\right)=\sum_{d}S_{cd}^{ba}(2\theta)\,R_{ca}^{d}\left(\frac{\mathrm{i}\pi}{2}+\theta\right)$$

Main question of this talk

How should we modify this boundary crossing relation ?

Our work [S.S , S.Yamaguchi, to appear]

Assumption

1. Bulk integrability
2. Boundary is "weakly-symmetric" under the non-invertible symmetry.

→ Boundary crossing relation must be also modified as

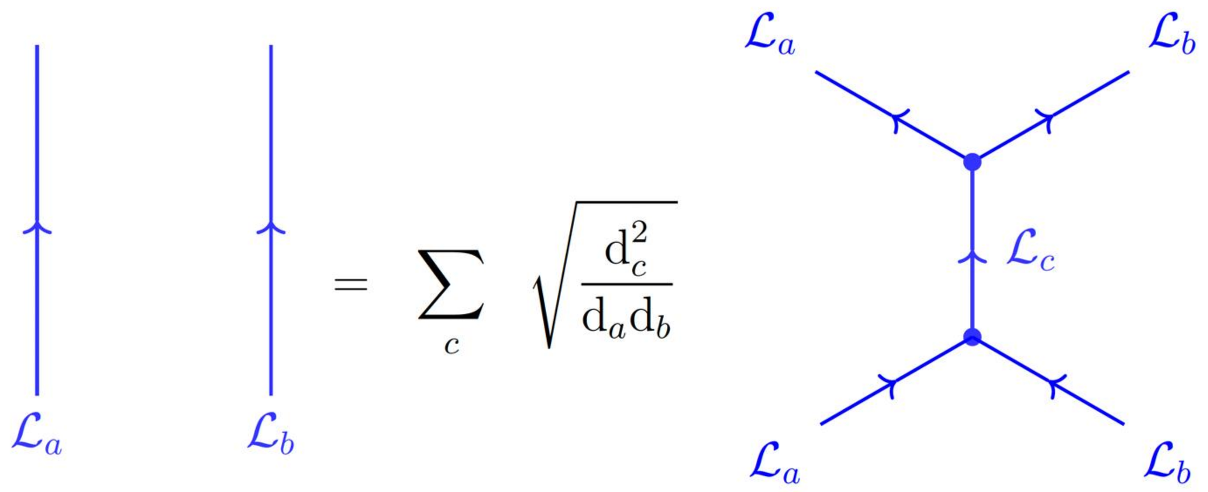
$$R^{b}{}_{ca}\left(\frac{\mathrm{i}\pi}{2}-\theta\right)=\sum_{d}\sqrt{\frac{\mathrm{d}_d}{\mathrm{d}_b}}\,S^{ba}_{cd}(2\theta)\,R^{d}{}_{ca}\left(\frac{\mathrm{i}\pi}{2}+\theta\right)$$

2. Non-invertible symmetries and kinks

- Non-invertible symmetry : non-group fusion rule

$$\mathcal{L}_a \otimes \mathcal{L}_b = \sum_c N^a_{bc}\, \mathcal{L}_c \ , \qquad N^a_{bc} \in \{0, 1\} \quad \text{(multiplicity-free case)}$$

- Partial fusion :



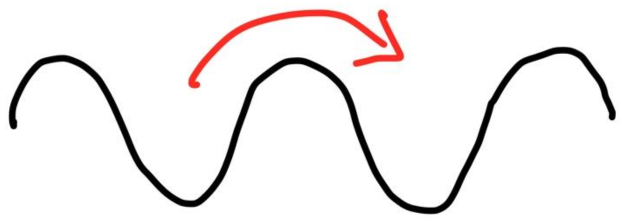
We assume that this fusion category is preserved along with RG flow to IR gapped phase.

Non-invertible symmetry is spontaneously broken in the IR.

→ We have degenerate vacua $|a\rangle$

$$|a\rangle = \mathcal{L}_a |0\rangle$$ (regular module category)

→ We can define the kink between $|a\rangle$ and $|a \pm 1\rangle$.



$|a\rangle$ $|a+1\rangle$

Ex.

Tricritical Ising CFT

$\downarrow$ $-\varepsilon'$

Gapped Phase
with 3 vacua

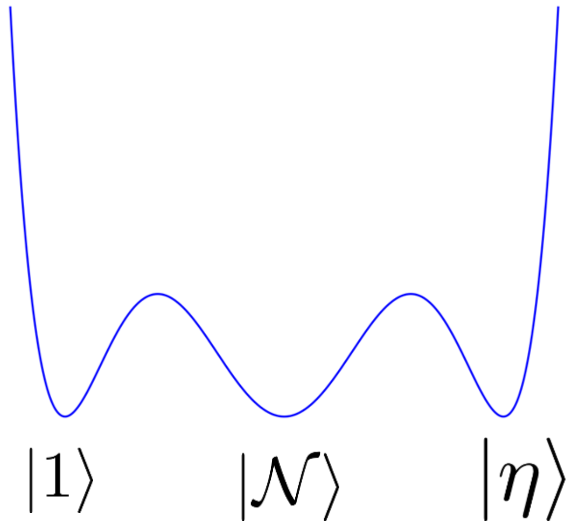
[Huse, ‘84; ……,
Chang, Lin, Shao, Wang, Yin, ‘19]

TY category is preserved

$$\mathcal{N} \otimes \mathcal{N} = 1 \oplus \eta$$

$$\mathcal{N} \otimes \eta = \eta \otimes \mathcal{N} = \mathcal{N}$$

$$\eta^2 = 1$$

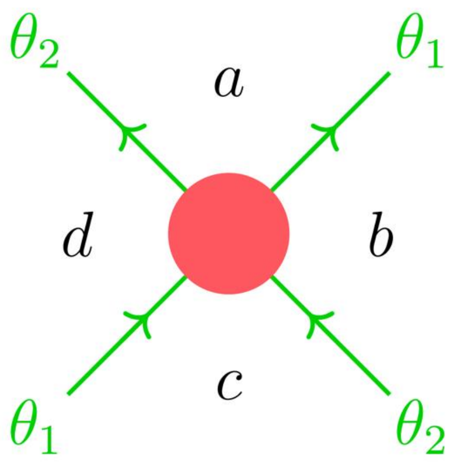


3. Boundary Scattering [Ghoshal, Zamolodchikov, '94]

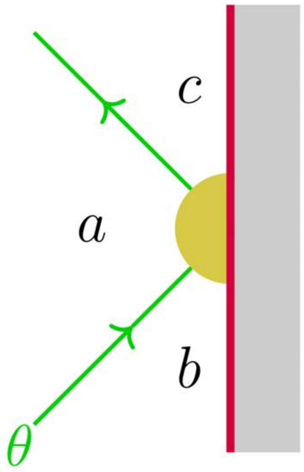
[Scattering data]

- Bulk S-matrix

not be affected by the boundary



- Boundary S-matrix



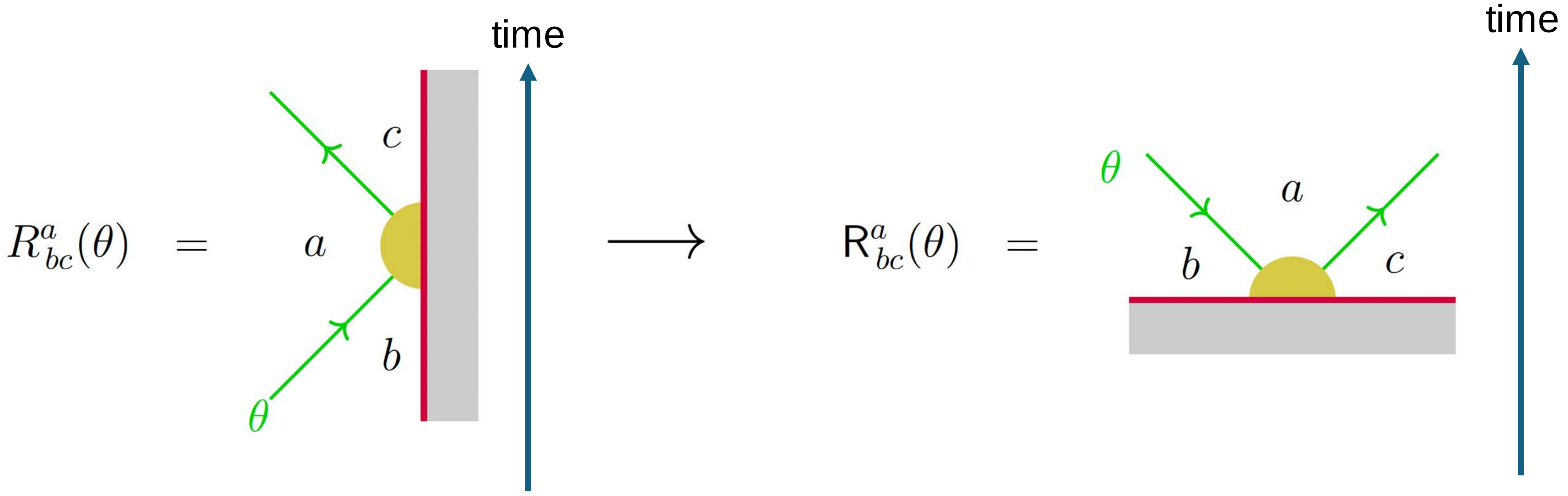
Recall that the bulk S-matrix must be subject to several conditions .

Ghoshal and Zamolodchikov proposed the three conditions for boundary S-matrix .

- Boundary crossing relation:

Boundary analog of crossing relation is more non-trivial.

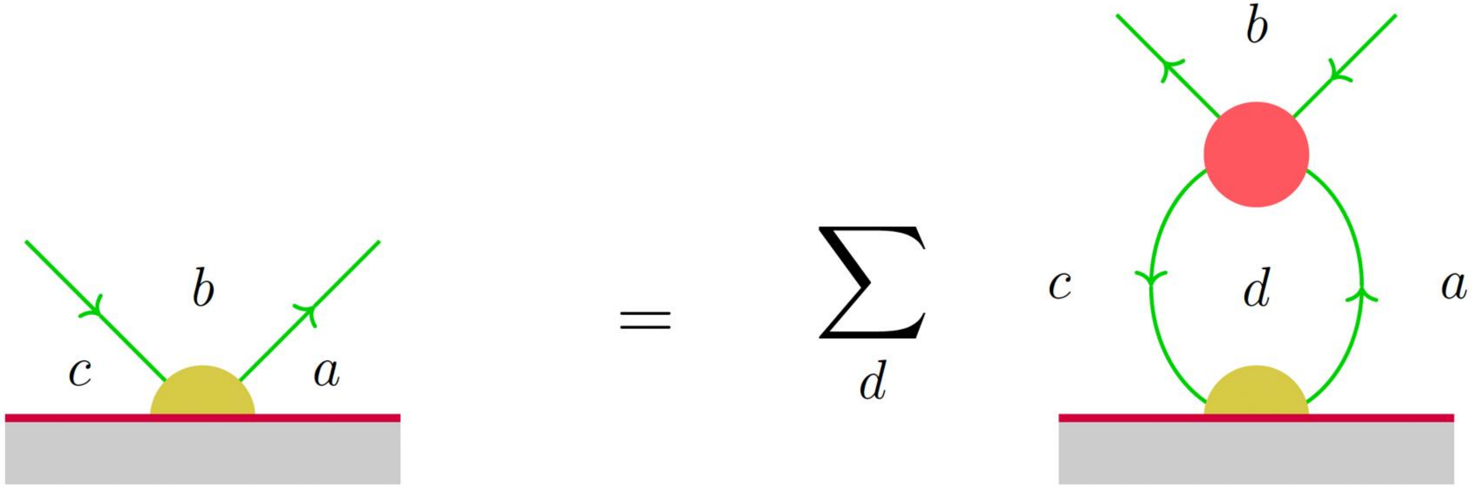
We first move to the rotated picture :



In this picture, the asymptotic kink state belongs to bulk Hilbert space.

In and out asymptotic states in the bulk Hilbert space must be related via bulk S-matrix .

$$\mathsf{R}^{b}_{\ ca}\left(\frac{\mathrm{i}\pi}{2}-\theta\right)=\sum_{d}S^{ba}_{cd}(2\theta)\,\mathsf{R}^{d}_{\ ca}\left(\frac{\mathrm{i}\pi}{2}+\theta\right)$$



Remark: we used the bulk integrability.

To sum up, boundary crossing relation can be put as follows:

$$\mathsf{R}^{a}_{bc}(\theta) = R^{a}_{bc}(\theta)$$

$$\mathsf{R}^{b}{}_{ca}\left(\frac{\mathrm{i}\pi}{2} - \theta\right) = \sum_{d} S^{ba}_{cd}(2\theta)\, \mathsf{R}^{d}{}_{ca}\left(\frac{\mathrm{i}\pi}{2} + \theta\right)$$

$$\longrightarrow \quad R^{b}{}_{ca}\left(\frac{\mathrm{i}\pi}{2} - \theta\right) = \sum_{d} S^{ba}_{cd}(2\theta)\, R^{d}{}_{ca}\left(\frac{\mathrm{i}\pi}{2} + \theta\right)$$

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??

→

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[Ghoshal, Zamolodchikov, ‘94]

Necessary to be modified when non-invertible symmetry is concerned .

4. Modified boundary crossing relation [S.S, S.Yamaguchi, to appear]

How should we modify ?

Our assumption :

1. Bulk integrability

4. Modified boundary crossing relation [S.S, S.Yamaguchi, to appear]

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How should we modify ?

Our assumption :

1. Bulk integrability → necessary to suppress multi-kink contributions

2. Boundary is weakly-symmetric under the non-invertible symmetry.

topological junction $\mathrm{Hom}(\mathcal{L}_a \otimes \mathcal{B}, \mathcal{B})$

[Choi, Rayhaun, Sanghavi, Shao, '23]

Firstly, we notice that in the rotated picture, the correctly normalized matrix R^{a}_{bc} must satisfy

$$\mathsf{R}^{b}_{ca}\left(\frac{\mathrm{i}\pi}{2}-\theta\right)=\sum_{d} S^{ba}_{cd}(2\theta)\,\mathsf{R}^{d}_{ca}\left(\frac{\mathrm{i}\pi}{2}+\theta\right)$$

But, we emphasize

$$\mathsf{R}^{a}_{bc}(\theta)\neq R^{a}_{bc}(\theta)$$

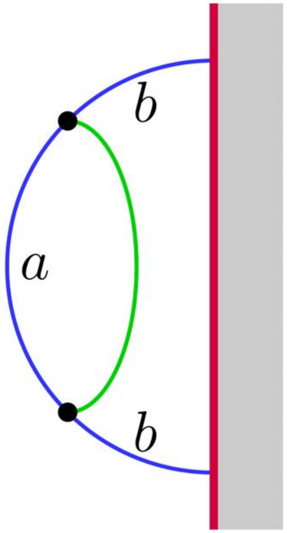
To find the proper relation, we must carefully look at the normalization!

- Normalization of $R^{a}_{bc}(\theta)$

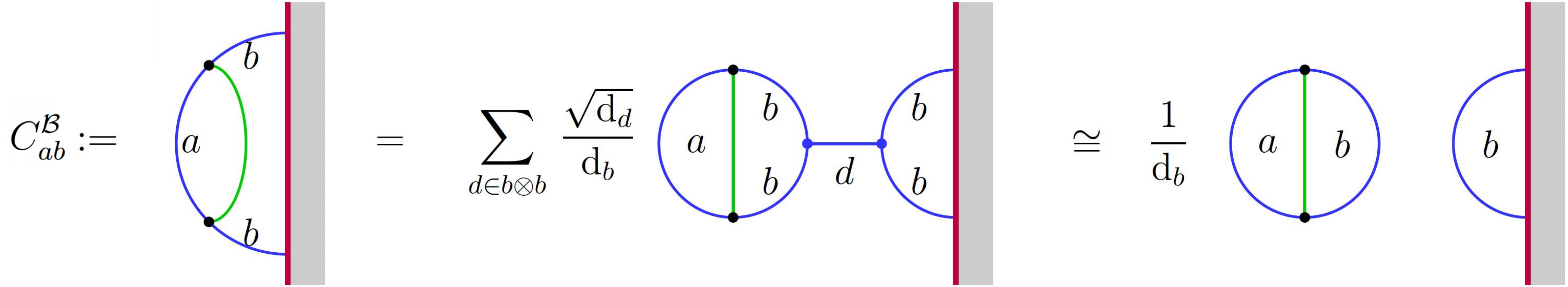
Correctly normalized boundary S-matrix is given by

$$R^{a}_{bc}(\theta) = \frac{1}{\sqrt{C^{\mathcal{B}}_{ac} \cdot C^{\mathcal{B}}_{ab}}} \, {}_{\text{out},\mathcal{B}}\langle K_{ac}(-\theta) | K_{ab}(\theta) \rangle_{\mathcal{B},\text{in}}$$

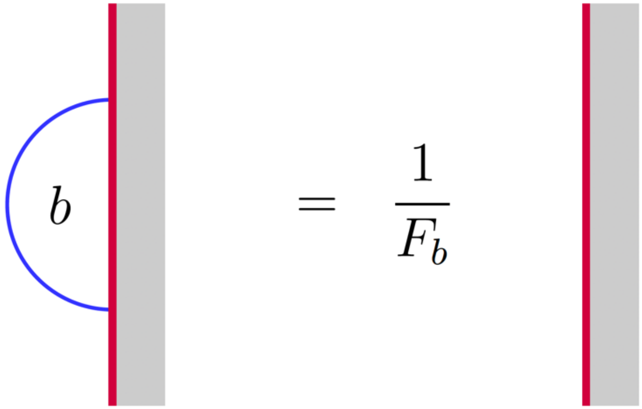
$$C^{\mathcal{B}}_{ab} := {}_{\mathcal{B}}\langle K_{ab}(\theta) | K_{ab}(\theta) \rangle_{\mathcal{B}} \quad =$$



By using the fusion property, we can evaluate as



Assuming



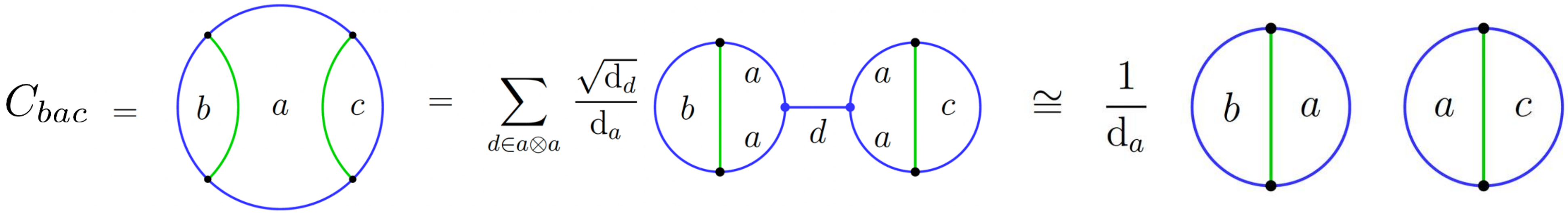
$$C^{\mathcal{B}}_{ab} = \frac{1}{G_b} \langle K_{ab}(\theta) | K_{ab}(\theta) \rangle \cdot {}_{\mathcal{B}}\langle 0 | 0 \rangle_{\mathcal{B}}$$

$(G_b := \mathrm{d}_b F_b)$

- Normalization of R^{a}_{bc}

Correctly normalized S-matrix in the rotated picture is given by

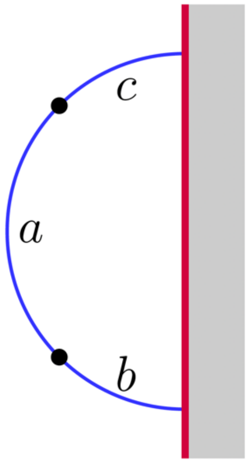
$$\mathsf{R}^{a}_{bc}(\theta) = \frac{1}{\sqrt{C_{bac} \cdot \langle \mathcal{B} | \mathcal{B} \rangle}} \, {}_{\rm out}\langle K_{ba}(-\theta) K_{ac}(\theta) | \mathcal{B} \rangle$$



$$\longrightarrow \quad C_{bac} = \frac{1}{\mathrm{d}_a} \langle K_{ba}(\theta) | K_{ba}(\theta) \rangle \, \langle K_{ac}(\theta) | K_{ac}(\theta) \rangle$$

By using the boundary LSZ formula and analytic continuation, we can relate the boundary S-matrices R^a_{bc} and R^a_{bc} in terms of two-point Euclidean correlators .

$$R^a_{bc}(\theta) = \frac{1}{\sqrt{C^{\mathcal{B}}_{ac}\cdot C^{\mathcal{B}}_{ab}}}$$



$$= \sqrt{\frac{C_{bac}\cdot\langle\mathcal{B}|\mathcal{B}\rangle}{C^{\mathcal{B}}_{ac}\cdot C^{\mathcal{B}}_{ab}}}\,\mathsf{R}^a_{bc} = \sqrt{\frac{G_b G_c}{\mathrm{d}_a}}\,\mathsf{R}^a_{bc}$$

$$\longrightarrow \quad \mathsf{R}^a_{bc}(\theta) = \sqrt{\frac{\mathrm{d}_a}{G_b G_c}}\,R^a_{bc}(\theta)$$

$$\mathsf{R}^{a}_{bc}(\theta) = \sqrt{\frac{\mathrm{d}_a}{G_b G_c}} R^{a}_{bc}(\theta)$$

Plug in

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$$\mathsf{R}^{b}_{ca}\left(\frac{\mathrm{i}\pi}{2} - \theta\right) = \sum_d S^{ba}_{cd}(2\theta)\, \mathsf{R}^{d}_{ca}\left(\frac{\mathrm{i}\pi}{2} + \theta\right)$$

Modified boundary crossing relation

$$R^{b}_{ca}\left(\frac{\mathrm{i}\pi}{2} - \theta\right) = \sum_d \sqrt{\frac{\mathrm{d}_d}{\mathrm{d}_b}}\, S^{ba}_{cd}(2\theta)\, R^{d}_{ca}\left(\frac{\mathrm{i}\pi}{2} + \theta\right)$$

5. Summary and prospects

- We revisit the boundary scattering theory.
- We found that the boundary crossing relation should be modified when non-invertible symmetry is concerned.

  (see our paper for concrete models)

- How about monopole scattering?

  [van Beest, Smith, Delmastro, Komargodski, Tong, ’23]

- Non-regular vacuum module category?