Symmetry defects in Maxwell theory without spin structure

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@ Non-perturbative methods in QFT, Kyushu University Institute for Advanced Study

Introduction

Symmetry = Topological defects

For an ordinary symmetry G:

- $\mathcal{D}_g(\Sigma_{d-1})$ is supported on a codim-1 mfd Σ_{d-1}.
- Conservation laws can be reproduced from topological nature.
- The fusion of the top. defects satisfies the group law: $\mathcal{D}_{g_1} \times \mathcal{D}_{g_2} = \mathcal{D}_{g_1 g_2}$.

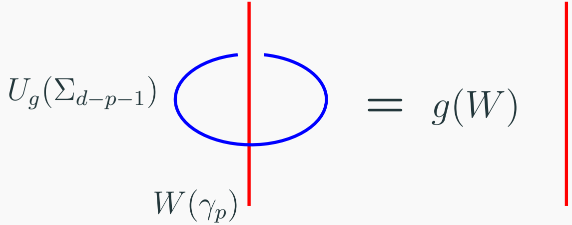
Higher-form symmetry

Topological defects with codim-1

$\Longrightarrow$ Topological defects with codim-$(p+1)$

[Gaiotto–Kapustin–Seiberg–Willett '14]

The p-form sym. is associated with a $(d-p-1)$-dim. top. defect $U_g(\Sigma_{d-p-1})$, which acts on a p-dim. object $W(\gamma_p)$ as



where Σ_{d-p-1} and γ_p are linked in spacetime, and $g(W)$ is a rep. of g.

1-form symmetries in Maxwell theory

4d Maxwell theory has two 1-form $\mathrm{U}(1)$ symmetries: $\mathrm{U}(1)_e^{[1]} \times \mathrm{U}(1)_m^{[1]}$.

The Euclidean action is

$$S = \frac{1}{2e^2} \int_{\mathcal{M}_4} F \wedge *F + \frac{\mathrm{i}\theta}{8\pi^2} F \wedge F.$$

The currents

$$*J_e = \frac{2}{e^2} * F, \qquad *J_m = \frac{1}{2\pi} F$$

are respectively conserved, $\mathrm{d} * J_e = 0$ and $\mathrm{d} * J_m = 0$, because of the EOM $\mathrm{d} * F = 0$ and Bianchi id. $\mathrm{d}F = 0$.

The 1-form symmetry $\mathrm{U}(1)_e^{[1]} \times \mathrm{U}(1)_m^{[1]}$ corresponds to the shift syms. for the photon $A \mapsto A + \lambda_e$, and dual photon $\tilde{A} \mapsto \tilde{A} + \lambda_m$, where $\alpha_{e,m} = \int \lambda_{e,m} \in \mathrm{U}(1)$.

The line ops., i.e., Wilson and 't Hooft line ops., are charged under $\mathrm{U}(1)_e^{[1]} \times \mathrm{U}(1)_m^{[1]}$:

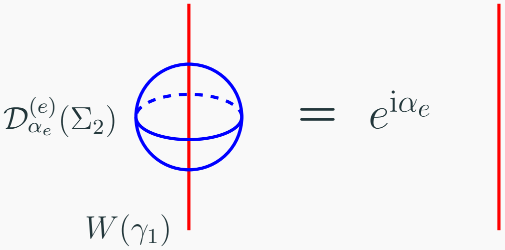
$$W^n(\gamma_1) := \exp\left(\mathrm{i}n \int_{\gamma_1} A\right) \mapsto e^{\mathrm{i}n\alpha_e} W^n(\gamma_1),$$
$$T^m(\gamma_1) := \exp\left(\mathrm{i}m \int_{\gamma_1} \tilde{A}\right) \mapsto e^{\mathrm{i}m\alpha_m} T^m(\gamma_1),$$

where $n, m \in \mathbb{Z}$ are the charges of the line ops.

The 1-form symmetry can be understood from the perspective of topological defects.

E.g., the topological defect for $\mathrm{U}(1)_e^{[1]}$ is

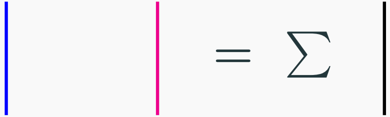
$$\mathcal{D}_{\alpha_e}^{(e)}(\Sigma_2) = \exp\left(\mathrm{i}\alpha_e \int_{\Sigma_2} \frac{2}{e^2} * F\right).$$



Non-invertible symmetry

Topological defects with group law

$\Longrightarrow$ Topological defects **without** group law



$$\mathcal{D}_a \quad \times \quad \mathcal{D}_b \quad = \sum_c N_{ab}^c \mathcal{D}_c$$

Particularly, they generally do **NOT** have an inverse such that

$$\mathcal{D}_a \times \mathcal{D}_a^{-1} = \mathcal{D}_a^{-1} \times \mathcal{D}_a = 1.$$

Recently, non-invertible symmetries in $d \geq 3$ are explored.

[Choi–Cordova–Hsin–Lam–Shao '21, Kaidi–Ohmori–Zheng '21,...]

Duality defects in Maxwell theory

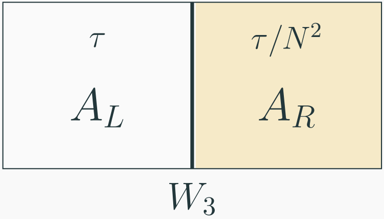
Gauge the magnetic $\mathbb{Z}_N^m \subset \mathrm{U}(1)_m^{[1]}$ in the Maxwell theory:

$$S = \int_{\mathcal{M}_4} \frac{1}{2e^2}|F|^2 + \frac{\mathrm{i}\theta}{8\pi^2}F^2 + \frac{\mathrm{i}}{2\pi}B_m F + \frac{\mathrm{i}N}{2\pi}B_m \mathrm{d}A_m,$$
$$= \int_{\mathcal{M}_4} \frac{N^2}{2e^2}|\mathrm{d}A_m|^2 + \frac{\mathrm{i}N^2\theta}{8\pi^2}\mathrm{d}A_m^2.$$

The magnetic $\mathbb{Z}_N^m$ gauging maps the coupling τ to $N^2\tau$, where $\tau := \theta/(2\pi) + 2\pi\mathrm{i}/e^2$.

Similarly, the electric $\mathbb{Z}_N^e$ gauging is $\tau \mapsto \tau/N^2$.

Let's electirically gauge $\mathbb{Z}_N^e$ in half of spacetime.



The interface W is topological.

The action of this system is

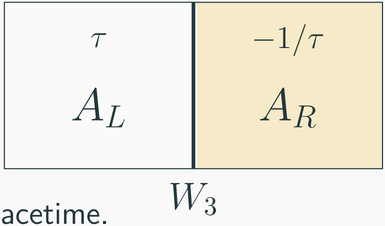
$$S = \int_{\mathcal{M}_L} \frac{1}{2e^2}|F_L|^2 + \frac{\mathrm{i}\theta}{8\pi^2}F_L^2 + \int_{\mathcal{M}_R} \frac{1}{2e^2}|F_R|^2 + \frac{\mathrm{i}\theta}{8\pi^2}F_R^2 + \frac{\mathrm{i}}{2\pi}\int_{W_3} a\,(N\,\mathrm{d}A_L - \mathrm{d}A_R),$$

where a is the 1-form $\mathrm{U}(1)$ gauge field defined only on W.

The last term is the interface action.

$\mathbf{SL}(2,\mathbb{Z})$ duality:

$$\mathbb{S} : \tau \mapsto -\frac{1}{\tau}\,, \quad \mathbb{T} : \tau \mapsto \tau + 1.$$



Perform the $\mathbb{S}$-transf. in half of spacetime.

The $\mathbb{S}$-interface on W is also topological. The action is

$$S = \int_{\mathcal{M}_L} \frac{1}{2e^2}|F_L|^2 + \frac{\mathrm{i}\theta}{8\pi^2}F_L^2 + \int_{\mathcal{M}_R} \frac{1}{2e^2}|F_R|^2 + \frac{\mathrm{i}\theta}{8\pi^2}F_R^2 + \frac{\mathrm{i}}{2\pi}\int_{W_3} A_L \mathrm{d} A_R.$$

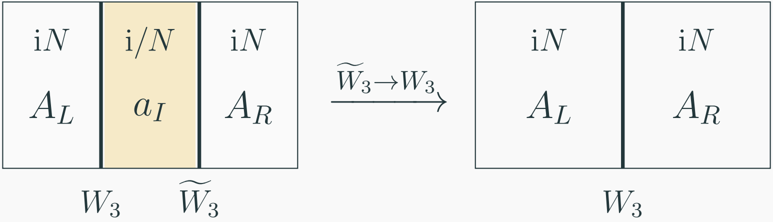
When $\tau = \mathrm{i}N$, the gauged theory returns to the original one:

$$\tau = \mathrm{i}N \xrightarrow{\text{gauge } \mathbb{Z}_N^e} \frac{\mathrm{i}}{N} \xrightarrow{\mathbb{S}\text{-transf.}} \tau = \mathrm{i}N.$$

Fusing the two interfaces, we obtain the top. defect:

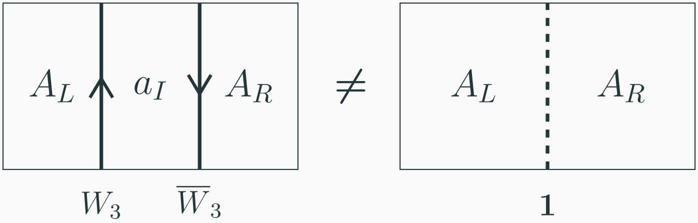
$$\frac{\mathrm{i}}{2\pi}\int_{W_3} a\left(N\,\mathrm{d}A_L - \mathrm{d}a_I\right) + \frac{\mathrm{i}}{2\pi}\int_{\widetilde{W}_3} a_I \mathrm{d}A_R$$

$$\xrightarrow{\widetilde{W}_3 \to W_3} \frac{\mathrm{i}N}{2\pi}\int_{W_3} A_L \mathrm{d}A_R$$



The topological defect $\mathcal{D}(W_3)$ is non-invertible:

$$\mathcal{D}_N(W_3) \times \mathcal{D}_N(\overline{W}_3) = \exp\left[\frac{\mathrm{i}N}{2\pi}\int_{W_3} a_I\,(\mathrm{d}A_L - \mathrm{d}A_R)\right] \neq \mathbf{1}.$$



Without spin structure

Maxwell theory on non-spin manifold

So far, we implicitly assumed $\mathcal{M}_4$ admits a spin structure.

Consider a orientable manifold that doesn't admit a spin structure (a non-spin manifold, e.g., $\mathbb{CP}^2$). Equivalently, the second SW class $w_2 \in H^2(\mathcal{M}_4, \mathbb{Z}_2)$ is nontrivial.

Then it is necessary to classify the possible spectrum of line operators [Aharony–Seiberg–Tachikawa '13], taking into account the spin-statistics of the line operators [Ang–Roumpedakis–Seifnashri '19].

We have four Maxwell theories depending on the spin-statistics of the charge-1 fundamental Wilson and 't Hooft lines:

	$W_\mathrm{b}T_\mathrm{b}$	$W_\mathrm{b}T_\mathrm{f}$	$W_\mathrm{f}T_\mathrm{b}$	$W_\mathrm{f}T_\mathrm{f}$
Wilson line W	**boson**	**boson**	**fermion**	**fermion**
't Hooft line T	**boson**	**fermion**	**boson**	**fermion**
Dyonic line WT	**fermion**	**boson**	**boson**	**fermion**

The all-fermion theory $W_\mathrm{f}T_\mathrm{f}$ has a pure gravitational anomaly, so we consider only $W_\mathrm{b}T_\mathrm{b}$, $W_\mathrm{b}T_\mathrm{f}$ and $W_\mathrm{f}T_\mathrm{b}$ below.

Fermionic line ops. are in the projective rep. of the spacetime symmetry $SO(4)$. Such line operators carrying electric/magnetic charge can be well-defined by coupling it to an appropriate bgd field of 1-form symmetry.

For example, $W_{\mathrm{f}}T_{\mathrm{b}}$ can be realized by choosing the bgd field of $\mathbb{Z}_2^e \subset \mathrm{U}(1)_e^{[1]}$ as $B_e = \pi w_2$. Then the fermionic fund. Wilson line dresses w_2 [Brennan–Córdova–Dumitrescu '22]:

$$W = e^{\mathrm{i}\int_{\gamma_1} A} \exp\left(\mathrm{i}\pi \int_{\Sigma_2} w_2\right),$$

where $\partial\Sigma_2 = \gamma_1$.

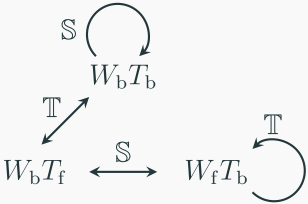
$\mathrm{SL}(2,\mathbb{Z})$ duality transformation

$\mathbb{S}$ and $\mathbb{T}$ act on the line operator $W^n T^m$ as, respectively,

$$\mathbb{S}:\ W^n T^m \mapsto W^m T^{-n}, \quad \mathbb{T}:\ W^n T^m \mapsto W^{n-m} T^m,$$

where the spin-statistics of the line operator do *not* change.

Therefore, the $\mathrm{SL}(2,\mathbb{Z})$ transformation does not close within a single theory and can be a map b/w different theories.



	$W_\mathrm{b}T_\mathrm{f}(\tau)$	$\to$	$W_\mathrm{f}T_\mathrm{b}(-1/\tau)$
	$(1,0)_\mathrm{b}$	$\mapsto$	$(0,-1)_\mathrm{b}$
$\mathbb{S}:$	$(0,1)_\mathrm{f}$	$\mapsto$	$(1,0)_\mathrm{f}$
	$(-1,0)_\mathrm{b}$	$\mapsto$	$(0,1)_\mathrm{b}$

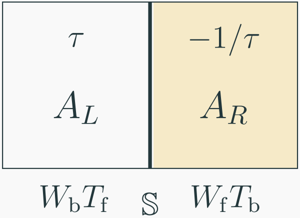
We can construct $\mathrm{SL}(2,\mathbb{Z})$-interfaces. For exmaple,

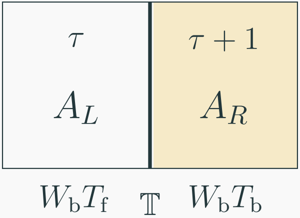
- $\mathbb{S}:\ W_{\mathrm{b}}T_{\mathrm{f}} \to W_{\mathrm{f}}T_{\mathrm{b}}$

$$\frac{\mathrm{i}}{2\pi}\int_{W_3} A_L\left(\mathrm{d}A_R + 2\pi w_2\right)$$

- $\mathbb{T}:\ W_{\mathrm{b}}T_{\mathrm{f}} \to W_{\mathrm{b}}T_{\mathrm{b}}$

$$\int_{W_3} \frac{\mathrm{i}}{2\pi} a\left(\mathrm{d}A_L - \mathrm{d}A_R\right) + \frac{\mathrm{i}}{4\pi} A_L \mathrm{d}A_R + \frac{\mathrm{i}}{2} A_L w_2$$





Gauging 1-form $\mathbb{Z}_2$ symmetries

We can also map b/w the theories by 1-form $\mathbb{Z}_2$ gauging.

The maps are characterized by TQFTs associated with the gauging.

E.g., gauge $\mathbb{Z}_2^m \subset U(1)_m^{[1]}$ in $W_\mathrm{b}T_\mathrm{b}$ as

$$S = \int_{\mathcal{M}_4} \frac{1}{2e^2}|F|^2 + \frac{\mathrm{i}\theta}{8\pi^2}F^2 + \frac{\mathrm{i}}{2\pi}B_m F + \frac{2\mathrm{i}}{2\pi}B_m \mathrm{d}A_m,$$
$$= \int_{\mathcal{M}_4} \frac{2^2}{2e^2}|\mathrm{d}A_m|^2 + \frac{\mathrm{i}2^2\theta}{8\pi^2}\mathrm{d}A_m^2.$$

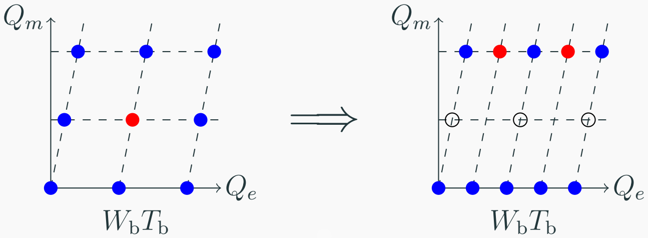
$\Longrightarrow$ This magnetic $\mathbb{Z}_2$-gauging maps $W_\mathrm{b}T_\mathrm{b}$ to $W_\mathrm{b}T_\mathrm{b}$.

The gauging map $W_\mathrm{b}T_\mathrm{b} \to W_\mathrm{b}T_\mathrm{b}$ can be interpreted from the perspective of the line operators.

The BF theory we coupled involves a bosonic line op. L:

$$S_{\mathrm{BF}} = \frac{2\mathrm{i}}{2\pi}\int_{\mathcal{M}_4} B_m A_m, \quad L = \exp\left(\mathrm{i}\int_{\gamma_1} A_m\right)$$

After gauging, L becomes a Wilson line with half-integral electric charge (in the sense of the original τ).



Alternatively, we perform $\mathbb{Z}_2^m$-gauging by coupling another TQFT:

$$S = \int_{\mathcal{M}_4} \frac{1}{2e^2}|F|^2 + \frac{\mathrm{i}\theta}{8\pi^2}F^2 + \frac{\mathrm{i}}{2\pi}B_m F + \frac{2\mathrm{i}}{2\pi}B_m\left(\mathrm{d}A_m + \pi w_2\right),$$

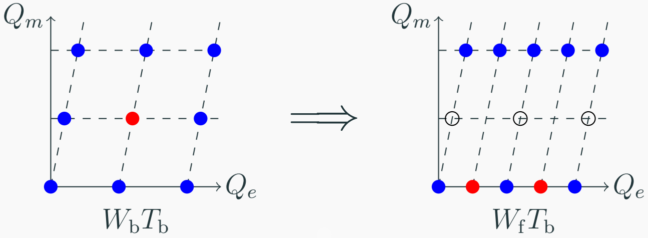
$$= \int_{\mathcal{M}_4} \frac{2^2}{2e^2}|F_m + \pi w_2|^2 + \frac{\mathrm{i}2^2\theta}{8\pi^2}(F_m + \pi w_2)^2.$$

$\Longrightarrow$ This magnetic $\mathbb{Z}_2$-gauging maps $W_\mathrm{b}T_\mathrm{b}$ to $W_\mathrm{f}T_\mathrm{b}$.

The TQFT with w_2 involves a fermionic line operator:

$$S_{\mathrm{BF}} = \frac{2\mathrm{i}}{2\pi} \int_{\mathcal{M}_4} B_m(\mathrm{d}A_m + \pi w_2), \quad L = e^{\mathrm{i}\int_{\gamma_1} A_m} \exp\left(\mathrm{i}\pi \int_{\Sigma_2} w_2\right)$$

Then after gauging, the Wilson operators with half-integral charge becomes fermionic.



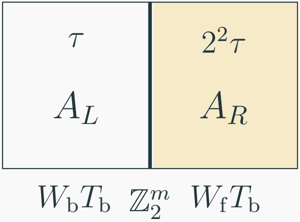
The gauging interfaces are, for example,

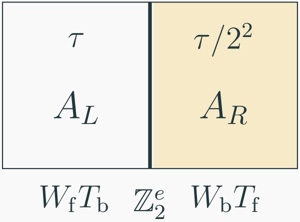
- $\mathbb{Z}_2^m: \; W_\mathrm{b}T_\mathrm{b} \to W_\mathrm{f}T_\mathrm{b}$

$$\frac{\mathrm{i}}{2\pi}\int_{W_3} a\left(\mathrm{d}A_L - 2\left(\mathrm{d}A_R + \pi w_2\right)\right)$$

- $\mathbb{Z}_2^e: \; W_\mathrm{f}T_\mathrm{b} \to W_\mathrm{b}T_\mathrm{f}$

$$\int_{W_3} \frac{\mathrm{i}}{2\pi} a\left(2\left(\mathrm{d}A_L + \pi w_2\right) - \mathrm{d}A_R\right) + \frac{\mathrm{i}}{2} A_R w_2$$





Topological defects in the non-spin Maxwell theory

For example, the $\mathbb{S}$ is an invertible symmetry in the spin Maxwell theory with $\tau = \mathrm{i}$ but not in the non-spin $W_\mathrm{b}T_\mathrm{f}$.

However, we can construct a topological defect by fusion $\mathbb{Z}_2^m$ and $\mathbb{Z}_2^e$-interfaces with $\mathbb{S}$-interface.

$$\int_{W_3} \frac{\mathrm{i}}{2\pi} a\,(\mathrm{d}A_L - 2\mathrm{d}b) + \frac{2\mathrm{i}}{2\pi} A_R\,\mathrm{d}b - \frac{\mathrm{i}}{2}(A_L - A_R)\,w_2,$$

i	$2^2\mathrm{i}$	i	i
A_R	a	b	A_R
$W_\mathrm{b}T_\mathrm{f}$	$W_\mathrm{b}T_\mathrm{b}$	$W_\mathrm{f}T_\mathrm{b}$	$W_\mathrm{b}T_\mathrm{f}$

In general, focusing on the subgroup $\mathbb{Z}_2^e \times \mathbb{Z}_2^m \subset \mathrm{U}(1)_e^{[1]} \times \mathrm{U}(1)_m^{[1]}$, we can classify topological defects that are allowed on the spin manifold but cannot exist on the non-spin manifold.

In other words, we identify mixed 't Hooft anomalies b/w gravity and non-invertible symmetries.

$\boldsymbol{W}_{\mathbf{b}}\boldsymbol{T}_{\mathbf{b}}$:

$$\pm\,\mathbb{S}^{-1}\cdot\mathbb{T}^{q}\cdot\mathbb{S}\cdot\mathbb{T}^{-p},\ \pm\,\mathbb{T}^{-1}\cdot\mathbb{S}^{-1}\cdot\mathbb{T}^{-1}\cdot\mathbb{S},\ \pm\,\mathbb{T}\cdot\mathbb{S}^{-1}\cdot\mathbb{T}\cdot\mathbb{S}$$

$(p,q)=(\text{even},\text{odd})\,,\ (\text{odd},\text{even})\,,\ (\text{odd},\text{odd}).$

$\boldsymbol{W}_{\mathbf{b}}\boldsymbol{T}_{\mathbf{f}}$:

$\pm\,\mathbb{S}^{-1}\cdot\mathbb{T}^{q}\cdot\mathbb{S}\cdot\mathbb{T}^{-p},\ \pm\,\mathbb{T}^{-1}\cdot\mathbb{S}^{-1}\cdot\mathbb{T}^{-1}\cdot\mathbb{S},\ \pm\,\mathbb{T}\cdot\mathbb{S}^{-1}\cdot\mathbb{T}\cdot\mathbb{S},\ \mathbb{S}^{\pm}$

$(p,q)=(\text{even},\text{even})\,,\ (\text{even},\text{odd}).$

$\boldsymbol{W}_{\mathbf{f}}\boldsymbol{T}_{\mathbf{b}}$:

$\pm\,\mathbb{S}^{-1}\cdot\mathbb{T}^{q}\cdot\mathbb{S}\cdot\mathbb{T}^{-p},\ \pm\,\mathbb{T}^{-1}\cdot\mathbb{S}^{-1}\cdot\mathbb{T}^{-1}\cdot\mathbb{S},\ \pm\,\mathbb{T}\cdot\mathbb{S}^{-1}\cdot\mathbb{T}\cdot\mathbb{S},\ \mathbb{S}^{\pm}$

$(p,q)=(\text{even},\text{even})\,,\ (\text{odd},\text{even}).$

Summary

We have the three types of non-anomalous Maxwell theories on the non-spin manifold. In this talk, we discussed symmetry, i.e., topological defects, in these theories.

To construct the symmetry defects, we constructed the top. interfaces of $\mathrm{SL}(2,\mathbb{Z})$ duality transf. and gauging, and fuse them.

We also classified symmetry defects that are allowed on the spin manifold but cannot exist on the non-spin manifold.