On the classification of duality defects in $c = 2$ compact boson CFTs with a discrete group orbifold

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Overview

We investigate duality symmetries of compact boson CFTs with $c = 2$.
For $c = 1$ case, many researchers have studied the case.(Thorngren Wan 2021, Ginsparg 1988,...)
$\rightarrow$Then we generalize it to $c = 2$ case and introduce some quadratic equations to classify them.
Moreover, for "almost all" cases, we can classify the solutions of the equations completely.

Outline

Motivation

- The notion of symmetry has been changed for the last decade
- Non-invetible ones and higher form symmetry
- For 2d CFT, the duality defects have been investigated by (Verlinde1988) or (Flölich,Fuchs,Runkel,Schweigert 20XX)...
- It is important to study what kind of duality symmetry 2d CFT have.
- It is easier to compute them for compact boson CFT.(Nagoya,Shimamori 2023, Damia et al. 2024, Thorngren,Wang 2021)

Basics of compact boson CFTs

Conformal field theory (CFT): Field theory that have conformal invariance (Critical Ising model, 3-state Potts model, $\mathcal{N} = 2$ 4d SYM,...)
We will mainly consider compact boson CFT which is defined as CFT constructed by n bosons compactified on n dim. torus $\mathbb{R}^n/\Lambda$ with rank 2 antisymmetric tensor B.
We can explicitly write the momentum (p_L, p_R) of a vertex operator $\exp(ip_L \cdot X_L + ip_R \cdot X_R)$ as

$$\begin{pmatrix} p_L \\ p_R \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \Lambda^* & (I+B)\Lambda \\ \Lambda^* & (I-B)\Lambda \end{pmatrix} \begin{pmatrix} N \\ W \end{pmatrix}$$

where N, W are n-elements integer vectors representing momentum nunbers and winding numbers respectively.

How to construct non-invertible symmetry?-half space gauging

Construction of non-invertible symmetry: Half-space gauging
Divide spacetime into two halves and apply orbifolding only to one half of the theory. If the result is equivalent to the original theory:

$$\mathcal{T} \simeq \mathcal{T}/G$$

The interface at the divided spacetime is referred to as a duality defect. Therefore, it is crucial that the theory is self-dual under the orbifolding.

Brief introduction to orbifold

Orbifold: An operation that creates a new theory by restricting the Hilbert space of a theory using its symmetry or by adding twisted sectors.

$$\mathcal{Z} = \sum_{[h,g]=0} \mathrm{Tr}_{\mathcal{H}_h} \left(g q^{L_0 - \frac{c}{24}} \bar{q}^{\bar{L}_0 - \frac{c}{24}} \right)$$

Gauging: An operation to obtain a gauge theory by realizing the symmetry of a theory as a gauge field.
$\rightarrow$ As a result of this operation, the theory generally transforms into a different one.

Global symmetry of $c = 1$ case

Let us consider an orbifold in the case of $c = 1$.
For $c = 1$: Compactify a boson on a circle of radius R. Hence, the only parameter of the theory is the radius R.
Here, let

$$\theta = \frac{1}{R}(X_L + X_R), \phi = R(X_L - X_R)$$

then the vertex operator can be written as $\exp(i(n\theta + w\phi))$.
Shifts of θ and ϕ: $U(1)_\theta \times U(1)_\phi$ symmetry.
In the following, we consider the theory obtained by orbifolding the discrete subgroup $\mathbb{Z}_N \times \mathbb{Z}_W$ of $U(1) \times U(1)$ symmetry (GCD(N, W)= 1).

$c = 1$case: When they are self-dual?

How to construct an orbifold theory: First, construct the Hilbert space invariant under the symmetry action.
For example, considering only $\mathbb{Z}_N$ symmetry... n is restricted to multiples of N.
Contribution of twisted sectors: w takes fractional values of the form $\frac{w'}{N}$.
Combining these results:

$$(p_L, p_R) = \left(\frac{n' N}{R} + \frac{w'}{N} R, \frac{n' N}{R} - \frac{w'}{N} R \right)$$

which matches the spectrum of a theory with radius $\frac{1}{N} R$.

Self-duality condition

Including $\mathbb{Z}_W$ symmetry, it can be shown that the orbifolded theory corresponds to a theory with radius $\frac{W}{N}R$.
T-Duality: $R \leftrightarrow \frac{1}{R}$

Self-duality condition

$$\frac{W}{N}R = \frac{1}{R}$$

For what radius R and choice of orbifold (i.e., (N, W)) does this condition hold?

$c = 1$ case: $\mathbb{Z}_k$ orbifold

As a simple example, choose a rational point $R = \sqrt{k}$ ($k \in \mathbb{Z}_{>0}$).
The problem above: What choice of N, W satisfies $\frac{W}{N}\sqrt{k} = \frac{1}{\sqrt{k}}$?
The answer is clear: Choose $N = k$ and $W = 1$.
$\rightarrow$ This yields the Tambara Yamagami category $\mathrm{TY}(\mathbb{Z}_k)$.

$c = 2$ case

What happens when extending the previous example to $c = 2$?
Metric of the compactified torus: $G : \Lambda^\top \Lambda$

New parameter: 2×2 antisymmetric matrix $\Lambda^\top B \Lambda = \begin{pmatrix} 0 & b \\ -b & 0 \end{pmatrix}$.

The degrees of freedom in these parameters are 4, making the moduli space 4-dimensional over $\mathbb{R}$.

$$\tau = \frac{G_{12}}{G_{11}} + \frac{i\sqrt{\det G}}{G_{11}}, \ \rho = b + i\sqrt{\det G}$$

These parameters describe the theory.

Parameters of orbifold

Number of components for θ and ϕ: Both increase to two. Consequently, the global symmetry $U(1)$ is expressed as a product of four factors.
Symmetry to be orbifolded: A diagonal subgroup $\mathbb{Z}_{N_1} \times \mathbb{Z}_{N_2} \times \mathbb{Z}_{W_1} \times \mathbb{Z}_{W_2}$ of $U(1)^4$.
Question: What theory results from such an orbifold? A clue lies in the explicit form of θ and ϕ:

$$\theta = \frac{\Lambda^{-1}}{\sqrt{2}}(X_L + X_R)$$
$$\phi = \frac{\Lambda^{\top}}{\sqrt{2}}((I - B)X_L - (I + B)X_R)$$

Parameters of orbifold

For simplicity, consider an orbifold by $\mathbb{Z}_{N_1}$ alone. This affects only θ_1 and ϕ_1.

$$\theta_1 \mapsto N_1\theta_1, \;\; \phi_1 \mapsto \frac{1}{N_1}\phi_1$$

Thus, the first column of Λ^{-1} should be scaled by $\frac{1}{N_1}$, leading to:

$$G_{11} \mapsto \frac{1}{N_1^2}G_{11}, G_{12} \mapsto \frac{1}{N_1}G_{12}, G_{22} \mapsto G_{22}$$

As a result:

$$\tau \mapsto N_1\tau, \;\; \rho \mapsto \frac{1}{N_1}\rho$$

Extending this to all components:

$$\tau \mapsto \frac{N_1W_2}{N_2W_1}\tau, \quad \rho \mapsto \frac{W_1W_2}{N_1N_2}\rho.$$

T-duality for $c = 2$

T-Duality: A transformation that preserves the spectrum of the theory, equivalent to $\mathrm{O}(2, 2, \mathbb{Z})$.
The action of T-duality on (τ, ρ) is given by:

$$\mathrm{P}(\mathrm{SL}(2, \mathbb{Z})_\tau \times \mathrm{SL}(2, \mathbb{Z})_\rho) \rtimes (\mathbb{Z}_2^M \times \mathbb{Z}_2^I)$$

with the following specific actions:

$$\mathrm{SL}(2, \mathbb{Z})_\tau \ni \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} : \quad \tau \mapsto \frac{\alpha\tau + \beta}{\gamma\tau + \delta}$$

and

$$\mathbb{Z}_2^M : (\tau, \rho) \mapsto (\rho, \tau), \quad \mathbb{Z}_2^I : (\tau, \rho) \mapsto (-\bar{\tau}, -\bar{\rho}).$$

Rewriting the condition for self-duality

In T-duality, there are two $\mathbb{Z}_2$ components, so the duality defect can be classified into four types based on their values (m, i).
Consider the simplest case: $(m, i) = (0, 0)$.
In this case, the condition for a theory (τ, ρ) to remain self-dual under the orbifold $\mathbb{Z}_{N_1} \times \mathbb{Z}_{N_2} \times \mathbb{Z}_{W_1} \times \mathbb{Z}_{W_2}$ is given by:

$$\frac{N_1 W_2}{N_2 W_1}\tau = \frac{\alpha\tau + \beta}{\gamma\tau + \delta}, \quad \frac{W_1 W_2}{N_1 N_2}\rho = \frac{\alpha'\rho + \beta'}{\gamma'\rho + \delta'}$$

where $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ is an elements of $\mathrm{SL}(2, \mathbb{Z})$ and so as $\begin{pmatrix} \alpha' & \beta' \\ \gamma' & \delta' \end{pmatrix}$.

Theorem for $(m, i) = (0, 0)$

The existence of $\alpha, \beta, \gamma, \delta, \alpha', \beta', \gamma,' \delta'$ that satisfy the equation above is equivalent to the following statement:
Two quadratic equations in x, y, x', y':

$$(N_2 W_1)^2 x^2 - 2N_1 N_2 W_1 W_2 (\Re\tau) xy + (N_1 W_2)^2 |\tau|^2 y^2 = N_1 N_2 W_1 W_2$$
$$(N_1 N_2)^2 x'^2 - 2N_1 N_2 W_1 W_2 (\Re\rho) x' y' + (W_1 W_2)^2 |\rho|^2 y'^2 = N_1 N_2 W_1 W_2$$

have integer solutions (x_0, y_0, x_0', y_0').

Theorem for $(m, i) = (0, 0)$: condition for intger-ness

In addition, given these solutions, the four numbers

$$z_0 = -\frac{N_1 W_2}{N_2 W_1}|\tau|^2 y_0, \quad w_0 = -2(\Re\tau) y_0 + \frac{N_2 W_1}{N_1 W_2} x_0,$$
$$z_0' = -\frac{W_1 W_2}{N_1 N_2}|\rho|^2 y_0', \quad w_0' = -2(\Re\rho) y_0' + \frac{N_1 N_2}{W_1 W_2} x_0'$$

are all integers.
In this case, the corresponding $\mathrm{SL}(2, \mathbb{Z})$ elements are:

$$\alpha = x_0,\ \beta = z_0,\ \gamma = y_0,\ \delta = w_0,$$
$$\alpha' = x_0',\ \beta' = z_0,\ \gamma' = y_0',\ \delta' = w_0'$$

which corresponds to the desired statement.

proof

By transforming the condition, we obtain a quadratic equation in x that τ satisfies:
Here, we use the following proposition:
Proposition: If a quadratic equation with real coefficients, $f(x) = 0$, has a complex root $x = z$, then $x = \bar{z}$ is also a root of the equation.
Thus, the equation above also has $\bar{\tau}$ as a solution. From the relationship between the roots and coefficients, by normalizing the leading coefficient to 1, the coefficients of the x-term and the constant term can be uniquely determined. Using these coefficient relations and transforming $\alpha\delta - \beta\gamma = 1$, we derive the equation that α and γ satisfy.
This derived equation is the desired self-duality condition.

Diferrent point and same point from $(m, i) = (0, 0)$

In the presence of mirror symmetry, it is not possible to reduce the equation satisfied by τ into a single-parameter equation.
$\rightarrow$ A relation that connects the two parameters τ and ρ is required.
How is this done? There exist real numbers p, q, t such that:

$$p\tau = q\rho + t$$

Using this relation, for example, τ can be expressed in terms of ρ. This results in two quadratic equations, each involving a single parameter!

Theorem for $(m, i) = (1, 0)$

The existence of $\alpha, \beta, \gamma, \delta, \alpha', \beta', \gamma,' \delta'$ that satisfy the equation (**??**) above is equivalent to the following statement:
Two quadratic equations in x, y, x', y':

$$p(N_2W_1)^2x^2 - 2pN_1N_2W_1W_2(\Re\tau)xy + p(N_1W_2)^2|\tau|^2y^2 = qN_1N_2W_1W_2$$
$$q(N_1N_2)^2x'^2 - 2qN_1N_2W_1W_2(\Re\rho)x'y' + q(W_1W_2)^2|\rho|^2y'^2 = pN_1N_2W_1W_2$$

have integer solutions (x_0, y_0, x'_0, y'_0). In addition, given these solutions, the four numbers

$$z_0 = \frac{t}{q}x_0 - \frac{pN_1W_2}{qN_2W_1}|\tau|^2y_0, \quad w_0 = -\left(2(\Re\rho) + \frac{t}{q}\right)y_0 + \frac{pN_2W_1}{qN_1W_2}x_0,$$
$$z'_0 = -\frac{t}{p}x'_0 - \frac{qW_1W_2}{pN_1N_2}|\rho|^2y'_0, \quad w'_0 = -\left(2(\Re\tau) - \frac{t}{p}\right)y'_0 + \frac{qN_1N_2}{pW_1W_2}x'_0$$

are all integers.

Brief example:$(\tau, \rho) = (i, i)$

Consider $(\tau, \rho) = (i, i)$, which corresponds to the tensor product of two $c = 1$ theories. The underlying theory is a compact boson at $R = 1$, possessing SU(2) symmetry. Using the parameters of this theory, quadratic equations can be constructed.
For example, in the case of $(m, i) = (0, 0)$:

$$(N_2 W_1)^2 x^2 + (N_1 W_2)^2 y^2 = N_1 N_2 W_1 W_2$$

This equation admits integer solutions only when $N_1 = N_2 = W_1 = W_2 = 1$, and the solutions (x, y) are limited to $(0, 1)$ and $(1, 0)$. For different values of (m, i), the conclusion remains entirely the same.

restriction to the existence of solutions

For $(m, i) = (0, 0)$, the only solutions to the equations are $(x, y) = (0, 1)$ or $(1, 0)$.
$\rightarrow$ This simplicity makes the classification of duality symmetries straightforward. In most cases, if solutions exist, either x or y must always be 0!

Proposition: Let r be a positive rational number. If τ is a representative of $\mathrm{SL}(2, \mathbb{Z})$ that is not $e^{\frac{2\pi i}{3}}$, then for the equation:

$$rx^2 - 2(\Re\tau)xy + \frac{1}{r}|\tau|^2 y^2 = 1,$$

the integer solutions, if they exist, are limited to cases where $x = 0$ or $y = 0$.

This result significantly simplifies the search for parameters N_1, N_2, W_1, W_2 that yield solutions. Since the equation's solutions are restricted to $x = 0$ or $y = 0$, finding valid configurations becomes much easier and reduces computational complexity.

Overview of the proof

We will focus on the case $\Re\tau > 0$ and on the solution for $x, y > 0$. We can set τ to $|\tau| \geq 1, |\Re\tau| \leq \frac{1}{2}$.
Assume that we have a integer solution (x, y) such that $xy \neq 0$ Then

$$\begin{aligned} 1 &= rx^2 - 2(\Re\tau)xy + \frac{1}{r}|\tau|^2 y^2 \\ &\geq 2(|\tau| - \Re\tau)xy \\ &> xy, \end{aligned}$$

which contradicts $xy \neq 0$.

Multicritical point

The moduli space of conformal field theories (CFTs) constructed from compact bosons consists of two branches: the toroidal branch and the orbifold branch. Their intersection points are referred to as multicritical points.

Example: KT Point in $c = 1$ CFT

The KT point is a multicritical point in $c = 1$ CFT. It corresponds to the theory of a compact boson at radius $R = 2$. Alternatively, it can be described as the theory of a boson at radius $R = 1$, orbifolded by $\mathbb{Z}_2$.

Key Feature

An exactly marginal deformation along the orbifold branch preserves the Tambara Yamagami category $\mathrm{TY}(\mathbb{Z}_4)$.

By calculating the duality symmetry at the multicritical point, we can deduce the symmetries preserved under deformation along the orbifold branch. This provides crucial insights into the structure and behavior of the theory near the multicritical point.

Duality symmetries at multicritical points

$(\tau, \rho) = (it, \frac{1}{2} + it)$ **with** $t \in \mathbb{Q}$
The two quadratic equations are

$$(N_2 W_1)^2 x^2 + t^2 (N_1 W_2) y^2 = N_1 N_2 W_1 W_2$$

$$(N_1 N_2)^2 x^2 - N_1 N_2 W_1 W_2 xy + \frac{1+4t^2}{4} (W_1 W_2) y^2 = N_1 N_2 W_1 W_2$$

If we assume $t \neq \frac{\sqrt{3}}{2}$, from the proposition, we can conclude that each of these equations has only two integer solutions: $(x, y) = (1, 0), (0, 1)$.
Choose one solution for each eq. $\rightarrow$Combine two conditions for $N_1, N_2, W_1, W_2$$\rightarrow$We can determine these numbers uniquely$\rightarrow$ Duality defect

Discussion

1. **Focus on $c = 2$ Compact Boson CFTs**: We investigated the existence of duality defects in $c = 2$ compact boson CFTs. The orbifold group was chosen as the diagonal subgroup of the $\mathrm{U}(1)^4$ symmetry: $\mathbb{Z}_{N_1} \times \mathbb{Z}_{N_2} \times \mathbb{Z}_{W_1} \times \mathbb{Z}_{W_2}$.
2. **Necessary Conditions for Duality Defects**: The necessary conditions for the existence of duality defects were expressed using the modular parameters. These conditions were classified into four cases and represented as a system of two quadratic equations.
3. **Simplified Solutions**: We demonstrated that in most cases, the solutions to these equations, if they exist, are limited to simple cases.
4. **Duality Symmetry at Multicritical Points**: The duality symmetry at several multicritical points was successfully computed.

Future works

1. **Generalization to Non-Diagonal Subgroups**: Extend the analysis to orbifolds by non-diagonal subgroups of $\mathrm{U}(1)^4$.
2. **Symmetries Preserved by Deformations**: Analyze which deformations preserve specific symmetries at multicritical points.
3. **Generalization to Arbitrary Central Charges**: Broaden the study to include CFTs with general central charges.