Noninvertible symmetry protected topological (SPT) phases in lattice models

Weiguang Cao

Arxiv: 2502.20435,

w/ Linhao Li (UGhent), Masahito Yamazaki (UTokyo & Kavli IPMU)

SDU University of Southern Denmark

QM

date: 2025/03/12

Outline

1. Motivation
2. Noninvertible symmetries in lattice models
3. Noninvertible SPTs
4. Conclusion

Symmetry are important!

— Symmetry means ***conservation***, Noether's theorem

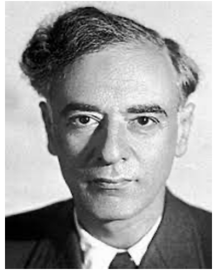
Every continuous symmetry of a system have a conservation law.

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \vec{j} = 0, \;\; Q = \int dx^3 \rho(x)$$

Flux conservation in free Maxwell theory → one-form $U(1)$ symmetry

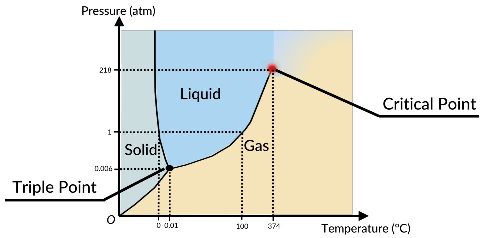
Conservation implies $[H, O_a] = 0$ → discrete symmetry

— Symmetry ***classifies*** phases, Landau's paradigm



Phases of matter are labelled by how they represent their symmetries.

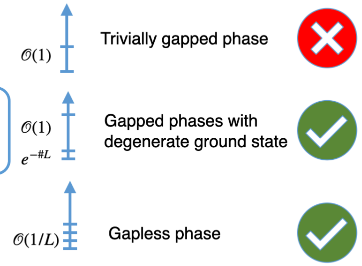
1. Preserving the symmetry
2. Spontaneously breaking the symmetry



— Symmetry ***constrains*** dynamics, 't Hooft anomaly

The macroscopic behavior is constrained by the anomaly of the symmetry.

If a model has a symmetry with ***'t Hooft anomaly***, the model cannot be trivially gapped.



(Youtube video:QFT Aspects of Symmetry - Ken Intriligator)

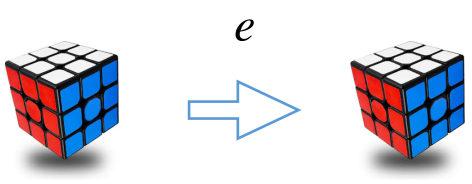
Symmetry with a group structure

Ordinary symmetry transformations behave like group elements

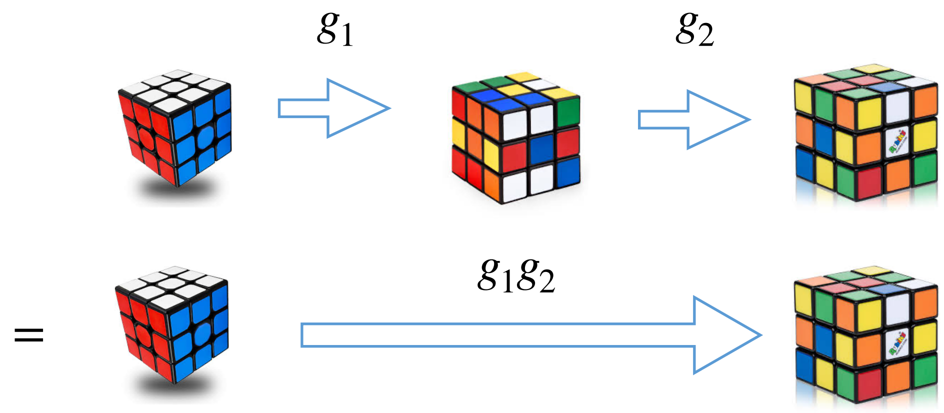
Can we have symmetry transformations beyond group?

For group G, its elements satisfies

1. Identity



2. Composition (fusion)



3. Inverse

Idea of non-invertible symmetry

Group → Fusion category

— Noninvertible symmetry are ubiquitous in CFT, TQFT, QFT and lattice systems

1. Imposing nontrivial selection rules
2. Finer classification of quantum phases of matter
   → generalized Landau paradigm
3. Constraints on RG flow and low energy dynamics

Symmetry and phases

Landau's paradigm: phases represent symmetries

New symmetries → new phases

In recently 20 years, we have gained more understanding on quantum phases of matters.

There exist exotic phases of matter ***beyond*** Landau's paradigm

In a $(1+1)d$ system with a given symmetry G:

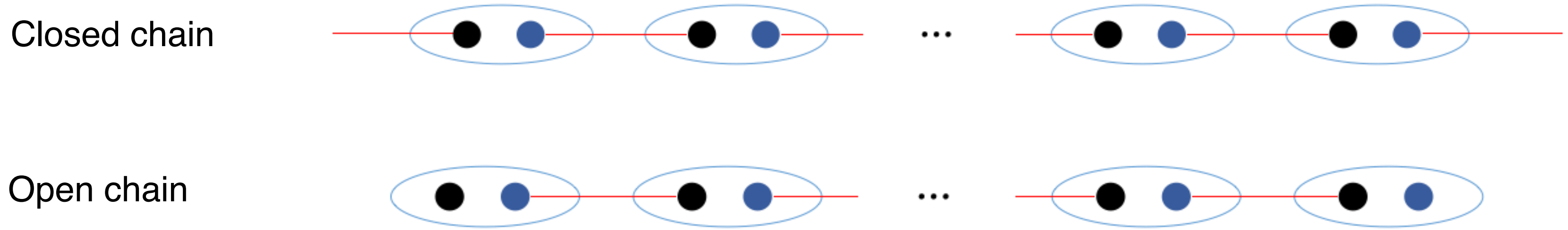
- Phases spontaneously breaking symmetry G
- Phases persevering symmetry G
  - Trivial (product) phases
  - Phases connected to the trivial phase only by ***breaking the symmetry***

The last case is also called ***symmetry protected topological (SPT) phases***.

SPT phases in 1+1d

— SPT phase is a gapped ***short-range-entangled*** phase with certain symmetry G

— On a closed chain, SPT phases are ***uniquely gapped***, while on a open chain, SPT phases exhibits ***edge modes*** at the boundary



— The edge modes are characterized by the ***projective representation*** of the symmetry operators at the boundaries

E.g. for $\mathbb{Z}_2 \times \mathbb{Z}_2$ SPT, when acting on the ground state, two $\mathbb{Z}_2$ doesn't commute at each boundary

SPT phases w/ group symmetry in general dimensions

—**Field**: G-SPT is an invertible theory, w/ anomaly inflow to the boundary.

—**Lattice**: G-SPT phase is a gapped phase w/ unique ground state with symmetry G on a closed manifold. W/ open boundary conditions, there exists anomalous edge modes on the boundary of the G-SPT.

—**Math**: G-SPT in $d+1$ dimension, or the anomaly of G symmetry in d dimension, is classified by group cohomology $H^{d+1}(G, U(1))$. 1106.4772, Chen, Gu, Liu & Wen

SPT phases w/ noninvertible symmetry

— Field: 1912.01817, Thorngren & Wang

— Lattice:

2312.09272, Fechisin, Tantivasadakarn & Albert, group-based cluster states

2404.01369, Seifnashri & Shao, Rep(D_8) SPT phases

2408.15960, Inamura & Ohyama, MPS setup

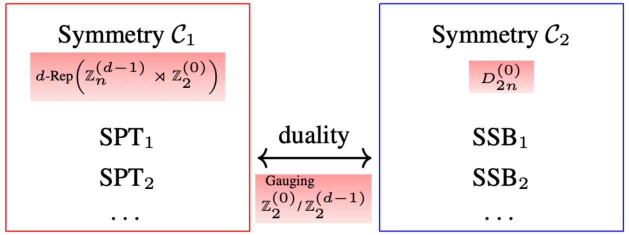
— Math: (Key word) "fibre functors of a fusion category"

We use ***duality/gauging*** method to study SPTs w/ noninvertible symmetry dual to D_{2n} symmetry in general $(d+1)$ dimensions.

— We use gauging non-normal $\mathbb{Z}_2^{(0)}$ subgroup as the duality transformation

— $d = 1$, the dual symmetry is Rep(D_{2n})

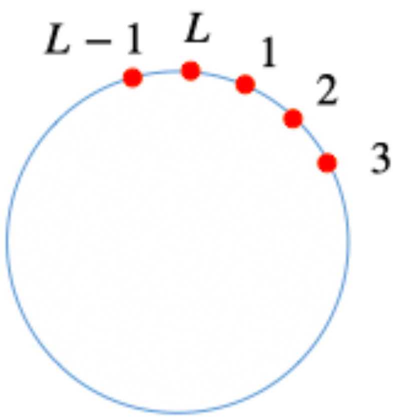
— for general d, the dual symmetry is conjectured to be d-Rep$\left(\mathbb{Z}_n^{(d-1)} \rtimes \mathbb{Z}_2^{(0)}\right)$



Outline

1. Motivation
2. Noninvertible symmetries in lattice models
3. Noninvertible SPTs
4. Conclusion

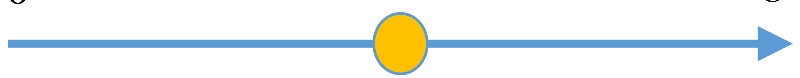
Noninvertible Kramers-Wannier duality



— Non-invertible symmetry can be found in phase transition

Quantum Ising model: $H(g) = -\sum_{i=1}^{L}\left(g^{-1}\ \sigma_i^z\sigma_{i+1}^z + g\ \sigma_i^x\right)$ With $\mathbb{Z}_2$ spin flip symmetry: $U = \prod_{j=1}^{L}\sigma_j^x$

Phase diagram:



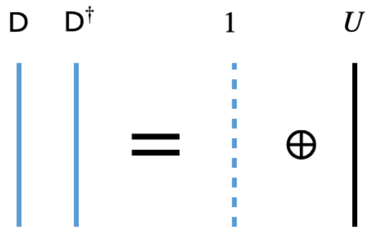
— Kramers-Wannier duality (KW) $\sigma_i^z\sigma_{i+1}^z$ D = D σ_i^x, σ_i^x D = D $\sigma_{i-1}^z\sigma_i^z$ $\rightarrow H(g)$ D = D $H(g^{-1})$

(Also understand as gauging $\mathbb{Z}_2$ spin flip symmetry)

— Transformation on states, D: SSB phase ↔ symmetric phase → D$|\uparrow\uparrow\ldots\uparrow\rangle$ = D$|\downarrow\downarrow\ldots\downarrow\rangle = |++\ldots+\rangle$

→D$|\psi\rangle = 0$, $|\psi\rangle = \left(|\uparrow\uparrow\ldots\uparrow\rangle - |\downarrow\downarrow\ldots\downarrow\rangle\right)/\sqrt{2}$ → ***KW is noninvertible!***

— Noninvertible symmetry at critical point. $H(1) = -\sum_{i=1}^{L}\left(\sigma_i^z\sigma_{i+1}^z + \sigma_i^x\right)$



a noninvertible symmetry!

Fusion rule: D × D$^\dagger = 1 + U$

Noninvertible symmetry from KW duality

— Noninvertible symmetry can be found in dual models after KW

Dual XXZ model: $H = -\sum_{i=1}^{L} \sigma^z_{i-1}\sigma^z_{i+1}(1+\sigma^x_i) + \Delta\ \sigma^x_i$

w/ $\mathbb{Z}_2^e \times \mathbb{Z}_2^o$ symmetry: $U^e = \prod_{k=1}^{L/2} \sigma^x_{2k},\ U^o = \prod_{k=1}^{L/2} \sigma^x_{2k+1}$

⇕ KW

XXZ model: $H = -\sum_{i=1}^{L} \sigma^x_{i-1}\sigma^x_{i+1} + \sigma^y_{i-1}\sigma^y_{i+1} + \Delta\ \sigma^z_{i-1}\sigma^z_{i+1}$

w/ $U(1)^\eta \rtimes \mathbb{Z}_2^U$ symmetry:

$\eta(\theta) = \prod_{j=1}^{L} \exp(i\theta(1-\sigma^z_j)),\ U = \prod_{j=1}^{L} \sigma^x_j$

— Rep(D_8) symmetry

D_8 has four one dimensional irreducible representations (irreps): $1,\ U^e,\ U^o,\ U^eU^o = U$

and a two dimensional irrep: $\mathrm{N}\left(\frac{\pi}{4}\right)$: $\sigma^z_{i-1}\sigma^z_{i-1} \leftrightarrow \sigma^z_{i-1}\sigma^x_i\sigma^z_{i-1},\ \sigma^x_i \leftrightarrow \sigma^x_i$

w/ fusion rule: $\mathrm{N}\left(\frac{\pi}{4}\right) \times \mathrm{N}\left(\frac{\pi}{4}\right) = 1 + U^e + U^o + U^eU^o$

— Sandwich construction $\quad \mathrm{N}\left(\frac{\pi}{4}\right) = \mathrm{D}\ \eta\left(\frac{\pi}{4}\right)\ \mathrm{D}^\dagger$

Noninvertible cosine symmetry

Convention: $\mathsf{N}(0) := 1 + U,\ \mathsf{N}\left(\frac{\pi}{2}\right) := U^e + U^o$

— Cosine symmetry $\mathsf{N}(\theta) = \mathsf{D}\ \eta(\theta)\ \mathsf{D}^\dagger$ w/ fusion rule: $\mathsf{N}(\theta_1) \times \mathsf{N}(\theta_2) = \mathsf{N}(\theta_1 + \theta_2) + \mathsf{N}(\theta_1 - \theta_2)$

— Symmetry in Dual XXZ model

Invertible $\mathbb{Z}_2^e \times \mathbb{Z}_2^o$ symmetry: $1,\ U^e,\ U^o,\ U$

Noninvertible cosine symmetry: $\mathsf{N}(\theta) = \mathsf{D}\ \eta(\theta)\ \mathsf{D}^\dagger$

KW ⇔

Symmetry in XXZ model

$U(1)^\eta \rtimes \mathbb{Z}_2^U$ symmetry:

$\eta(\theta),\ U$

— Cosine symmetry contains Rep(D_{2n}) symmetry

$D_{2n} = \mathbb{Z}_n^\eta \rtimes \mathbb{Z}_2^U$ symmetry:

$\eta\left(\frac{\pi}{n}\right),\ U$

E.g. $D_6 = S_3$ has 2 one dimensional irreps: $1,\ U$

and a two dimensional irrep: $\mathsf{N}\left(\frac{\pi}{3}\right)$

$\mathsf{N}(\theta) = \mathsf{N}(\pi - \theta)$

w/ fusion rule: $\mathsf{N}\left(\frac{\pi}{3}\right) \times \mathsf{N}\left(\frac{\pi}{3}\right) = 1 + U + \mathsf{N}\left(\frac{\pi}{3}\right)$

Noninvertible cosine symmetry in general dimension

— Dual symmetry in $(d+1)$ dimensions

Higher-category structure $\begin{cases} \text{Invertible } \mathbb{Z}_2^{(d-1)} \times \mathbb{Z}_2^{(0)} \text{ symmetry: } W_\gamma \,,\, \eta\left(\frac{\pi}{2}\right) \\ \text{Noninvertible 0-form cosine symmetry: } \mathsf{N}(\theta) \end{cases}$ $\overset{\text{KW}}{\Longleftrightarrow}$ $U(1)^\eta \rtimes \mathbb{Z}_2^U$ symmetry

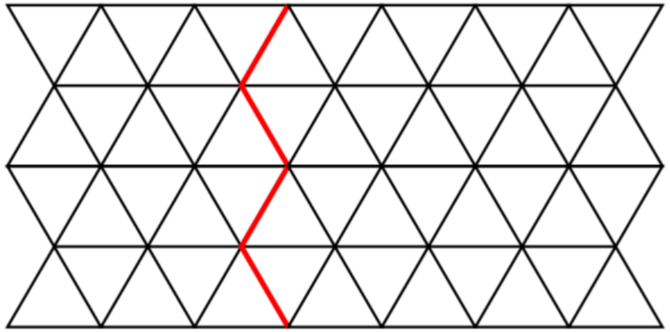
fusion rule: $\mathsf{N}\left(\theta_1\right) \times \mathsf{N}\left(\theta_2\right) = \mathsf{N}(\theta_1 + \theta_2) + \mathsf{N}(\theta_1 - \theta_2)$

$$\mathsf{N}(0) := 2\mathsf{C},\ \mathsf{C} = \frac{1}{2^{|\Sigma|}} \prod_{\gamma \in \Sigma} (1 + W_\gamma)$$

$$\mathsf{N}\left(\frac{\pi}{2}\right) := 2\mathsf{C}\ \eta\left(\frac{\pi}{2}\right)$$

— We are interested in discrete sub-symmetry, dual to 0-form D_{2n} symmetry

Realization of dual symmetry on lattice



– D_{2n} symmetry in $(2+1)$ dimensions

Consider the model on a general lattice Γ

Hilbert space: $\bigotimes_{v\in\Gamma}(\mathbb{C}^2\otimes\mathbb{C}^n)_v$, operators: $\sigma_v^x,\ \sigma_v^z,\ X_v,\ Z_v$

Symmetry operator: $\eta\left(\frac{\pi}{n}\right)=\prod_{v\in\Gamma}X_v,\ U=\prod_{v\in\Gamma}C_v\sigma_v^x$

– Gauging $\mathbb{Z}_2^U$ symmetry

1. Coupling the theory to $\mathbb{Z}_2$ gauge field μ_l^x,μ_l^z on the link

2. Imposing the Gauss law constraints $\left(G_v=C_v\sigma_v^x\prod_{l\ni v}\mu_l^z=1\right)$ to project onto the gauge invariant sector

e.g. $\sigma_i^z\sigma_j^z\to\sigma_i^z\mu_{<i,j>}^x\sigma_j^z\to\mu_{<i,j>}^x$

1-form symmetry $W_\gamma=\prod_{l\in\gamma}\mu_l^x$

$$\eta\left(\frac{\pi}{n}\right)\to \mathrm{N}\left(\frac{\pi}{n}\right)=\mathrm{C}\left(\prod_{v\in\Gamma}X_v^{\prod_{l\in P_{(v_0,v)}}\mu_l^x}+(\mathrm{h.c.})\right)$$

0-form noninvertible symmetry

Outline

1. Motivation
2. Noninvertible symmetries in lattice models
3. Noninvertible SPTs
4. Conclusion

General method

?

Known

Duality $\mathscr{D}$

Theories with symmetry $\mathscr{C}_1$ $\Longleftrightarrow$ Theories with symmetry $\mathscr{C}_2$

— Gapped phases of symmetry $\mathscr{C}_1$ and that of symmetry $\mathscr{C}_2$ are ***one-to-one***

Example: If $\mathscr{D}$ is *KW* (D : $\sigma_i^z\sigma_{i+1}^z \to \sigma_i^x, \quad \sigma_i^x \to \sigma_{i-1}^z\sigma_i^z$), and $\mathscr{C}_1 = \mathscr{C}_2 = \mathbb{Z}_2$

$$H_{sym} = -\sum_{i=1}^{L} \sigma_i^x \quad \leftarrow \text{KW} \rightarrow \quad H_{SSB} = -\sum_{j=1}^{L} \sigma_j^z \sigma_{j+1}^z$$

—Classification (construction) of the gapped phases of $\mathscr{C}_2$ (a group symmetry) implies the **classification (construction) of the gapped phases of** $\mathscr{C}_1$, in particular for $\mathscr{C}_1$ **SPT phases**

General method

Category

KW

Group

Theories with symmetry $\mathscr{C}_1$ $\longleftrightarrow$ Theories with symmetry $\mathscr{C}_2$

— Symmetry $\mathscr{C}_1$ (with a $\mathbb{Z}_2$ sub-symmetry) SPT → the KW dual theory has symmetry $\mathscr{C}_2$ but is $\mathbb{Z}_2$ SSB

— Classification of unbroken part of $\mathscr{C}_2$ → ***classification of the SPT phases for*** $\mathscr{C}_1$

—This method holds generally if $\mathscr{C}_2$ contains a 0-form $\mathbb{Z}_2$ sub-symmetry

— For group $\mathscr{C}_2$, It ***translates the problems of fusion category to that of group***

Example: Rep(D_8) SPT phases in $(1+1)d$

KW

Rep(D_8): N $\left(\frac{\pi}{4}\right)$, $1, U^e, U^o, U$ $\longleftrightarrow$ D_8: $1, \eta\left(\frac{\pi}{4}\right), \eta\left(\frac{\pi}{2}\right), \eta\left(\frac{3\pi}{4}\right), U$

Rep(D_8) SPT phases: $\mathbb{Z}_1 \oplus \mathbb{Z}_2$

Unbroken symmetry:

1. $\mathbb{Z}_4^{\eta}: \ H^2(\mathbb{Z}_4, U(1)) = \mathbb{Z}_1$
2. $\mathbb{Z}_2^{\eta^2} \times \mathbb{Z}_2^{\eta U}: \ H^2(\mathbb{Z}_2 \times \mathbb{Z}_2, U(1)) = \mathbb{Z}_2$

Trivial phase: $H = -\sum_{j=1}^{L} \sigma_j^x$

$\mathbb{Z}_1 : H = -\sum_{j=1}^{L} \sigma_j^z \sigma_{j+1}^z$

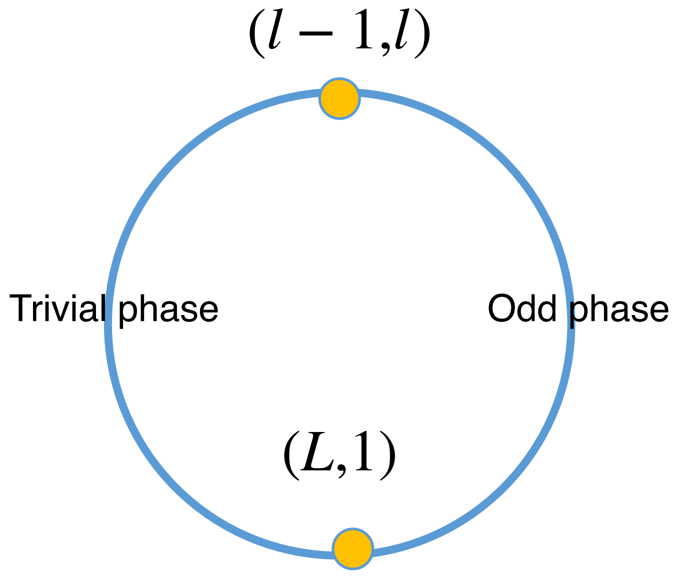
Odd SPT: $H_{odd} = \sum_{k=1}^{L/2} \sigma_{2k}^x + \sigma_{2k-1}^z \sigma_{2k}^x \sigma_{2k+1}^x \sigma_{2k+2}^x \sigma_{2k+3}^z + \sigma_{2k-1}^z \sigma_{2k+1}^x \sigma_{2k+3}^z$

$\mathbb{Z}_2 : H = \sum_{k=1}^{L/2} \sigma_{2k-1}^z \sigma_{2k}^z + \sum_{l=0}^{3} \eta^{-l} \left(\sigma_{2k-1}^y \sigma_{2k}^x \sigma_{2k+1}^x \sigma_{2k+2}^y \right) \eta^l$

Even SPT

The other phase is given by one site lattice translation

Edge modes of non-invertible SPT phases



— For example, consider the two distinct Rep(D_8) SPT phases on a closed chain

— Acting on the GS on a closed chain with two interfaces, symmetry operators ***factorizes***

$$U_e = 1, \quad U_o = \sigma^y_{l+1}\sigma^y_{L-1} = U^{\mathsf{L}}_o U^{\mathsf{R}}_o$$

$$N = \sigma^x_{l+1}\sigma^z_{L-1} + \sigma^z_{l+1}\sigma^x_{L-1} = N^{\mathsf{L},1}N^{\mathsf{R},1} + N^{\mathsf{L},2}N^{\mathsf{R},2}$$

— The edge modes come from the projective representation

$$U^{\mathsf{L}}_o N^{\mathsf{L},i} = -N^{\mathsf{L},i}U^{\mathsf{L}}_o, \quad U^{\mathsf{R}}_o N^{\mathsf{R},i} = -N^{\mathsf{R},i}U^{\mathsf{R}}_o, \quad \text{for } i = 1,2$$

— This analysis can be generalized to Rep(D_{2n}) SPTs in $(1+1)d$

Noninvertible SPTs in $(d+1)$ dimensions

— $d=1$: Rep(D_{2n}) SPTs $\to \mathbb{Z}_2^U$ SSB with $D_{2n} = \mathbb{Z}_n^\eta \rtimes \mathbb{Z}_2^U$

(1) $n = 1 \text{ mod } 2$, unbroken is $\mathbb{Z}_n^\eta$, $\to H^2(\mathbb{Z}_n, U(1)) = \mathbb{Z}_1$

(2) $n = 0 \text{ mod } 2$, unbroken is (i) $\mathbb{Z}_n^\eta$ or (ii) $\mathbb{Z}_{n/2}^{\eta^2} \rtimes \mathbb{Z}_2^{\eta U}$, $\to H^2(\mathbb{Z}_n, U(1)) \oplus H^2(\mathbb{Z}_{n/2} \rtimes \mathbb{Z}_2, U(1)) = \mathbb{Z}_1 \oplus \mathbb{Z}_{gcd(n/2,2)}$

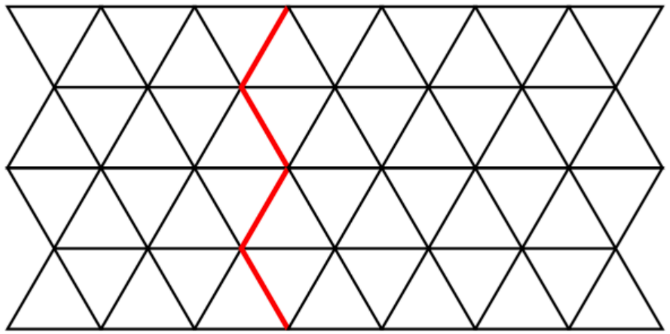
— For general d, the classification of dual noninvertible SPT is

(1) $n = 1 \text{ mod } 2$, $H^{d+1}(\mathbb{Z}_n, U(1))$;

(2) $n = 0 \text{ mod } 2$, $H^{d+1}(\mathbb{Z}_n, U(1)) \oplus H^{d+1}(\mathbb{Z}_{n/2} \rtimes \mathbb{Z}_2, U(1))$

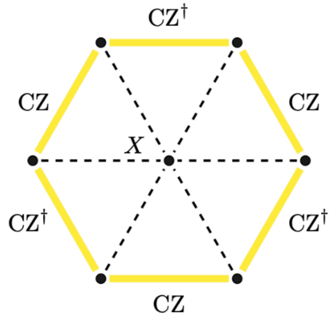
	$d=1$	$d=2$	$d=3$
$n = 1 \text{ mod } 2$	$\mathbb{Z}_1$	$\mathbb{Z}_n$	$\mathbb{Z}_1$
$n = 2 \text{ mod } 4$	$\mathbb{Z}_1 \oplus \mathbb{Z}_1$	$\mathbb{Z}_n \oplus (\mathbb{Z}_{n/2} \times \mathbb{Z}_2)$	$\mathbb{Z}_1 \oplus \mathbb{Z}_1$
$n = 0 \text{ mod } 4$	$\mathbb{Z}_1 \oplus \mathbb{Z}_2$	$\mathbb{Z}_n \oplus (\mathbb{Z}_{n/2} \times \mathbb{Z}_2 \times \mathbb{Z}_2)$	$\mathbb{Z}_1 \oplus (\mathbb{Z}_2 \times \mathbb{Z}_2)$

Examples of noninvertible SPTs in $(2+1)d$



– $\mathbb{Z}_2$ SSB with symmetry D_8,

classified by $\mathbb{Z}_4 \oplus (\mathbb{Z}_2 \otimes \mathbb{Z}_2 \otimes \mathbb{Z}_2)$



$$H_m = -\sum_{\langle i,j\rangle\in\Gamma} \sigma_i^z \sigma_j^z - \sum_{i\in\Gamma} X_i \left(\prod_{(i,j,k)\in\triangle} \mathrm{CZ}_{j,k} \prod_{(i,j,k)\in\triangledown} \mathrm{CZ}_{j,k}^\dagger \right)^m + (\mathrm{h.c.}), \quad m=0,1$$

$\updownarrow$ KW

– SPTs with 2-Rep$\left(\mathbb{Z}_4^{(1)} \rtimes \mathbb{Z}_2^{(0)}\right)$ symmetry

$$H_m^{\mathrm{SPT}} = -\sum_{\langle i,j\rangle} \mu^x_{\langle i,j\rangle} - \sum_i X_i \left(\prod_{(i,j,k)\in\triangle} \mathrm{CZ}^\mu_{i,j,k} \prod_{(i,j,k)\in\triangledown} (\mathrm{CZ}^\mu_{i,j,k})^\dagger \right)^m + (\mathrm{h.c.}), \quad m=0,1$$

Outline

1. Motivation

2. Noninvertible symmetries in lattice models

3. Noninvertible SPTs

4. Conclusion

Summary and outlook

1. We propose a general method to study gapped phases with noninvertible symmetry.
2. We get a full classification of SPT with noninvertible symmetry dual from D_{2n} symmetry in general dimensions.
3. We constructed and analyzed the Rep(D_{2n}) SPTs in $(1+1)d$ and examples of 2-Rep$\left(\mathbb{Z}_4^{(1)} \rtimes \mathbb{Z}_2^{(0)}\right)$ SPTs in $(2+1)d$.
4. A complete study of noninvertible SPTs in $(2+1)d$ can be studied in the future, e.g. construction the 2-Rep$\left(\mathbb{Z}_4^{(1)} \rtimes \mathbb{Z}_2^{(0)}\right)$ SPTs with classification $\mathbb{Z}_4 \oplus (\mathbb{Z}_2 \otimes \mathbb{Z}_2 \otimes \mathbb{Z}_2)$. The anomalous edges modes at the interface of different SPT phases could be interesting.

Nothing in physics seems so hopeful to as the idea that it is possible for a theory to have a high degree of symmetry was hidden from us in everyday life.

The physicist’s task is to find this deeper symmetry.

Thank you!