Shell Operator, ETH and Black Hole Information Paradox

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Non-perturbative methods in QFT
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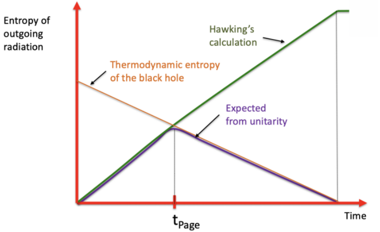
String Theory and Duality

- Pursuit quantum gravity: String theory $\&$ BH entropy $\&$ Information paradox.
- Quantum gravity seems to need UV completion of Einstein gravity to reduce degree of freedom, and only known UV completion is string theory (Swampland Program).
- String theory revolution: 5 consistent perturbative superstring theory, 11D supergravity, stringy description of $\frac{1}{2}$ BPS D_p brane, string duality.
- AdS/CFT: Gauge/Gravity duality, BH evaporation should be unitary. But in what way?

BH Information Paradox

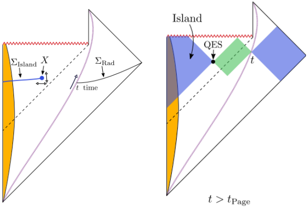
- Bekenstein and Hawking

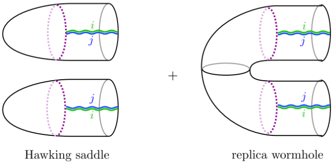


- BH complementarity: whether there is a firewall near behind black hole horizon [Almheiri,, Marolf, Polchinski, Sully]
- AdS/CFT version: late time Lorentzian behavior of two point correlator in thermal and finite boundary CFT [Maldacena]
- One simplified version: Euclidean behavior of thermal two point function [Fitzpatrick, Kaplan, Li and Wang]

Island and Ensemble Average

- Ryu-Takayanagi formula, fined grained entropy, Island formula
- Wormhole saddle





- Factorization problem: when Euclidean wormhole are included, semiclassical partition function no longer factorized

$$Z_{grav}[M_1 \cup M_2] \neq Z_{grav}[M_1] Z_{grav}[M_2]$$

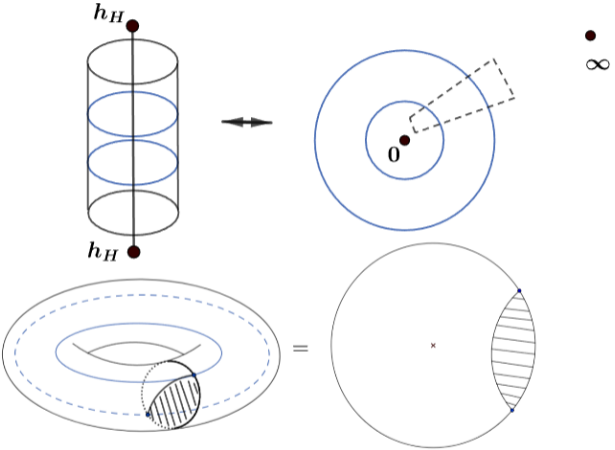
- Gravity/Average CFT: average over boundary CFT or high energy states above BH threshold?

AdS_3/CFT_2

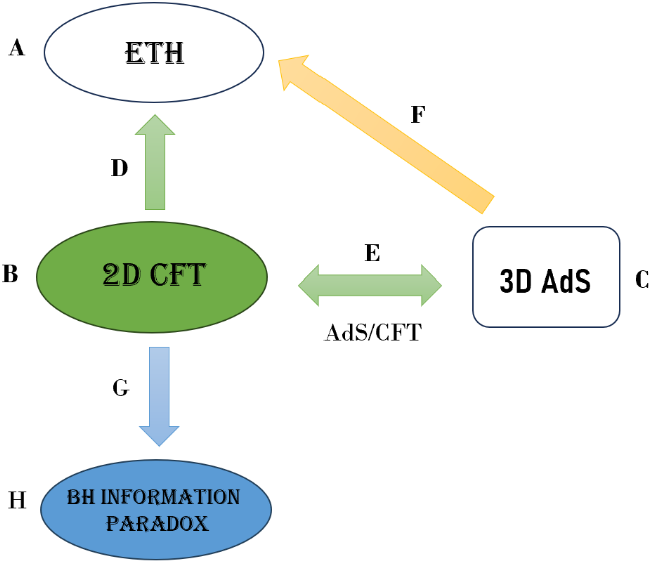
- More like dynamic gravity, curvature singularity; Under much better control than in other dimensions.
- A proposed exact stringy correspondence: strings on $AdS_3 \times S^3 \times T^4$ with one unit of NS-NS flux = $\mathrm{Sym}^N(T^4)$ (symmetry product theory) [Gaberdiel, Gopakumar, Eberhardt] .
- 3D Gravity: no propagating degree of freedom, maximally symmetric spacetime, non-trivial BTZ black hole.
- 2D CFT: Virasoro symmetry, RCFT models, modular invariance.

Our Setup

Goal: Using field theory techniques to calculate two point function of non-local shell operator.



Connection



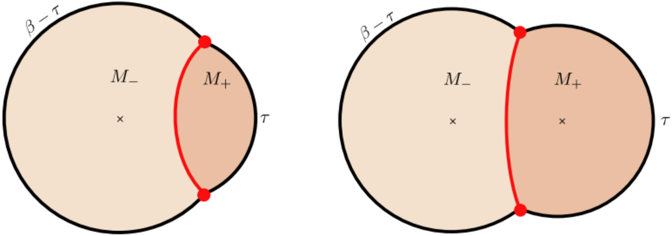
AdS$_3$ backreaction

- Quantity: thin-shell thermal correlator in $AdS_3 \longrightarrow$ non-trivial backreaction
- Process: thin shell of dust start from spacelike infinity and dive into Euclidean section of bulk
- Solution: saddle point solution of Euclidean action

$$I = -\frac{1}{16\pi G}\int \sqrt{g}(R - 2\Lambda) + \int_{\mathcal{W}} \sqrt{h}\sigma$$

AdS$_3$ backreaction

- Can be solved exactly with relations



$$\frac{r_-}{r_*} = \sin\frac{r_-(\beta-\tau)}{2}\,, \qquad \frac{r_+}{r_*} = \sin\frac{r_+\tau}{2}$$

- Thermal two-point function:

$$G_\beta(t) \approx e^{-\Delta I}\,, \qquad \Delta I = I + \log Z(\beta)$$

ETH analysis

- ETH assume matrix elements of thin shell operators in high energy spectrum are

$$O_{nm} \approx f(E_n)\delta_{nm} + e^{-S(\bar{E})/2} g(\bar{E},\omega)^{1/2} R_{nm}$$

- Need justification: Shell operators are heavy, completely delocalize over spatial manifold, however simple.

- Micro-canonical vs. Canonical ensemble: Shell operators are heavy, micro-canonical ensemble result, backreaction in the bulk.

ETH analysis

- Canonical and Micro-canonical relation:

$$\partial_\tau \log \mathcal{Z}_{\beta^*} G^2_{\beta^*}(\tau) = \partial_\tau \log G^2_{E_H}(\tau)$$

- Physical meaning: up to some multiplicative factor, canonical correlator equal to micro-canonical one at special mass and micro-canonical correlator equal to canonical one at special temperature.

- Gravity "like" or only "know" canonical correlator, so we need start from $Z_\beta G_\beta(\tau)$ to get $G_{M_m}(\tau)$

$$G_{E_H}(\tau) = \int d\beta\, e^{-S(E_H)+\beta E_H} \mathcal{Z}_\beta G_\beta(\tau) \sim e^{-S(E_H)+\beta^* E_H} \mathcal{Z}_{\beta^*} G_{\beta^*}(\tau)$$

where $\beta^* = \beta^*(E_H, \tau)$ are determined by following condition

$$E_H + \partial_\beta \log \mathcal{Z}_\beta G_\beta(\tau)|_{\beta=\beta^*} = 0$$

CFT_2: Virasoro Block

- Four point function $\langle HHLL \rangle$: thermal two local point correlator in Euclidean black hole background.

- Virasoro Block: group operators by Virasoro representation theory, approximation at large c

  Zamolodchikov

$$F(h_i, h_p, c, z) \approx e^{-\frac{c}{6} f(\frac{h_i}{c}, \frac{h_p}{c}, z)}$$

- Monodromy method and accessory parameter:

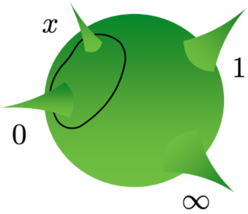
$$\chi''(z) + T(z)\,\chi(z) = 0, \quad \frac{\partial f(z)}{\partial z} = c_2(z)$$

where

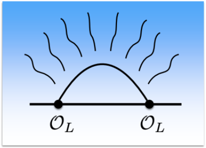
$$T(z) = \frac{h_1}{z^2} + \frac{h_2}{(z-z_0)^2} + \frac{h_3}{(z-1)^2} + \frac{h_1+h_2+h_3-h_4}{z(1-z)} + \frac{c_2(z) z_0 (1-z_0)}{z(z-z_0)(1-z)}$$

Some Intuitions

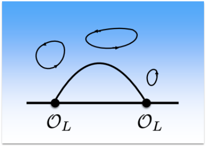
- Monodromy method:



- Two corrections:



$\epsilon_L \propto h_L/c$



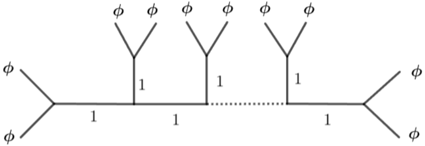
$1/c$

Continuum Limit Of Shell Operator

- Non-local $\rightarrow$ hard?
  Uniform non-local Shell operator $\rightarrow$ easier!

- Continuum limit of stress tensor

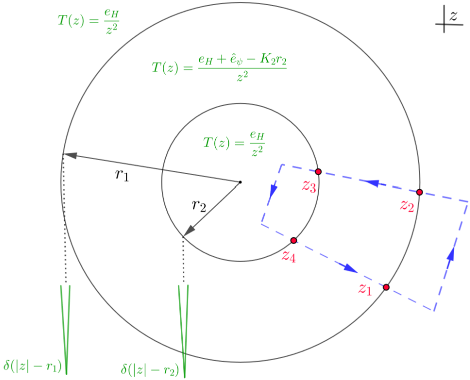
$$T(z) = \frac{h_H}{z^2} - \frac{c_H}{z} + \sum_{j=1}^{2} \sum_{i=1}^{n} \frac{h_\psi}{(z - z_j^{(i)})^2} - \frac{c_{\psi,i}}{z - z_j^{(i)}}$$



Singularity Become More Smooth

Singularities become more smooth in each finite region, but not globally holomorphic anymore

$$T(z,\bar{z}) = \frac{(\hat{e}_\psi - K_1 r_1)\Theta(|z|-r_1) + (\hat{e}_\psi - K_2 r_2)\Theta(|z|-r_2)}{z^2} - \frac{\hat{e}_\psi[r_1\delta(|z|-r_1) + r_2\delta(|z|-r_2)]}{z^2}$$



Vacuum Virasoro Block Solution

Monodromy matrix

$$M = J_2(z_1)J_2^{-1}(z_2)J_1(z_3)J_1^{-1}(z_4)$$

Monodromy equation

$$r_2^{2\rho_2}(\rho_2+\rho_1-\hat{e}_\psi)(\rho_2-\rho_1-\hat{e}_\psi) = r_1^{2\rho_2}(\rho_2-\rho_1+\hat{e}_\psi)(\rho_1+\rho_2+\hat{e}_\psi)$$

where

$$\hat{e}_\psi = ne_\psi\,, \quad \rho_1 = \frac{\sqrt{1-4e_H}}{2}, \quad \rho_2 = \frac{\sqrt{1-4(e_H+\hat{e}_\psi-K_2 r_2)}}{2}$$

This has a wide range of validity! Non-perturbative in $\frac{h_p}{c}$, but perturbative in $1/c$.

Result: ETH of non-local operator

- Probe limit:

$$
\begin{aligned}
\log G_{e_H} =& c_s(e_H, \hat{e}_\psi) - \frac{2c\hat{e}_\psi}{3} \log \sinh \frac{\sqrt{1-4e_H}t}{2} \\
&+ \frac{c\hat{e}_\psi^2}{3} \coth \frac{\sqrt{1-4e_H}t}{2} \left(t \coth \frac{\sqrt{1-4e_H}t}{2} - \frac{2}{\sqrt{1-4e_H}} \right) \\
&+ O(\hat{e}_\psi^3)
\end{aligned}
$$

- Keep terms up to the linearized order, the result is just the thermal two-point function in CFT_2, which is expected in the probe limit.
- This implies that ETH is also held by non-local simple shell operators in two dimensional holographic CFT!

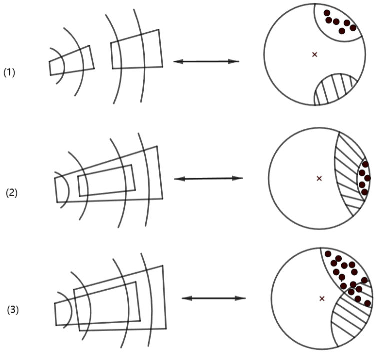
Result: Multi-shell and Gaussian Ensemble

- Leading order field theory results match Gauss ensemble

$$\langle E_H|\mathcal{S}_b(\tau_4)\mathcal{S}_b^\dagger(\tau_3)\mathcal{S}_a(\tau_2)\mathcal{S}_a^\dagger(\tau_1)|E_H\rangle,$$

$$\langle E_H|\mathcal{S}_b(\tau_4)\mathcal{S}_b^\dagger(\tau_3)\mathcal{S}_a(\tau_2)\mathcal{S}_a^\dagger(\tau_1)|E_H\rangle,$$

$$\langle E_H|\mathcal{S}_b(\tau_4)\mathcal{S}_b^\dagger(\tau_3)\mathcal{S}_a(\tau_2)\mathcal{S}_a^\dagger(\tau_1)|E_H\rangle,$$



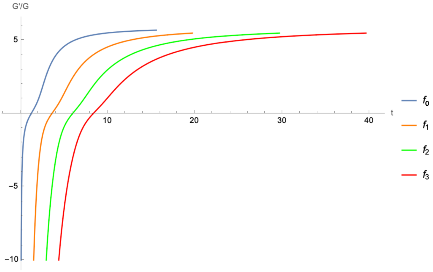
Result: Analytic Property I

- Singular at infinite number of points

$$t_l = \frac{2l\pi}{\sqrt{4e_H - 1}}, \quad l \in \mathbb{Z}$$

- Even though the perturbative solutions have forbidden singularities term by term, they all can be resolved non-perturbatively by summing over full probe corrections. [Kaplan, Faulkner, Huajia...]

- Exact in $\frac{h_p}{c}$, resummation automatically



Outcome: Analytic property II

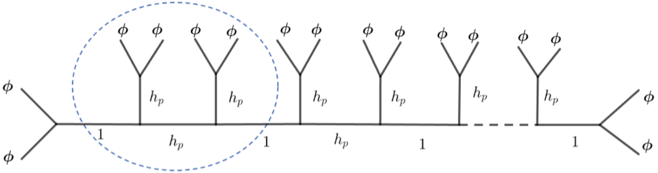
- Information loss problem $\rightarrow$ Periodicity in Euclidean correlator $\rightarrow$ Infinitely many forbidden singularities.
- What is the Lorentzian behavior?

(1): $$\frac{G'_{e_H}}{G_{e_H}} = -\frac{c\hat{e}_\psi\sqrt{1-4e_H}}{3}\coth\frac{\sqrt{1-4e_H}t}{2} + \frac{c\hat{e}_\psi^2[3+\cosh(\sqrt{1-4e_H}t) - 2\sqrt{1-4e_H}t\coth\frac{\sqrt{1-4e_H}t}{2}]}{3(\cosh(\sqrt{1-4e_H}t)-1)} + O(\hat{e}_\psi^3)$$

(2): $$\frac{G'_{e_H}}{G_{e_H}} = \frac{c[(4e_H-1)t^2-4k^2\pi^2]}{12t^2} + \frac{16k^2\pi^2c\hat{e}_\psi}{3(4e_H-1)t^3-12k^2\pi^2 t} - \frac{64k^2\pi^2c[3(4e_H-1)t^2+4k^2\pi^2]\hat{e}_\psi^2}{3[(4e_H-1)t^2-4k^2\pi^2]^3} + O(\hat{e}_\psi^3)$$

Future directions

- How to prove it? Sum over infinite kinematical data...
- Future problem: other blocks? need change the accessory equation...



- What is the meaning of Island in CFT language? Lorentzian lesson from Island?

Backup: Junction condition and bulk solution

- Consider the general problem of gluing two Euclidean Schwarzshild-AdS regions with local geometry

$$ds^2_\pm = f_\pm(r)dt^2_\pm + \frac{dr^2}{f_\pm(r)} + r^2 d\phi^2$$

where blackening factors are

$$f_\pm(r) = r^2 - 8GM_\pm$$

- Consider the pair $(h^\pm_{ab}, K^\pm_{ab})$ of induced metrics and extrinsic curvatures, define $\Delta h_{ab} = h^+_{ab} - h^-_{ab}$ and $\Delta K_{ab} = K^+_{ab} - K^-_{ab}$, then junction condition states that

$$\Delta h_{ab} = 0, \quad \Delta K_{ab} - h_{ab}\Delta K = -8\pi G T^W_{ab}$$

Backup: ETH analysis

For the micro-canonical correlator $G_{E_H}(\tau)$, we have

$$G_{E_H} = \langle E_H|\mathcal{S}(\tau)\mathcal{S}^+(0)|E_H\rangle = \sum_m \langle E_H|e^{\tau H}\mathcal{S}(0)e^{-\tau H}|E_m\rangle\langle E_m|\mathcal{S}^+(0)|E_H\rangle$$
$$= \sum_m e^{\tau(E_H-E_m)-f(E_H,E_m)}|R_{E_H,E_m}|^2 \sim \int dE\, e^{S(E)+\tau(E_H-E)-f(E_H,E)}$$
$$\sim e^{S(M_p)+\tau(E_H-M_p)-f(E_H,M_p)}$$

where $M_p = M_p(E_H,\tau)$ are determined by following condition

$$\left(\partial_E S(E) - \tau - \partial_E f(E_H,E)\right)\Big|_{E=M_p} = 0$$

The physical meaning of is that to compute micro-canonical correlator we need determine one effective "mass" using saddle point approximation to replace one infinite sum.

The End

Questions & Comments