On Energy and Entropy of Surface Defect in Six Dimensions

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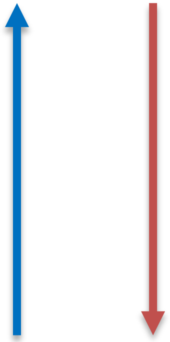
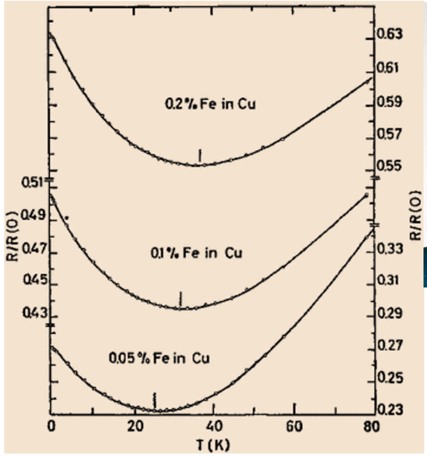
[with Ma-Ke Yuan, arXiv:2310.02096][with Zixiao Huang and Ma-Ke Yuan, arXiv:2501.09498]

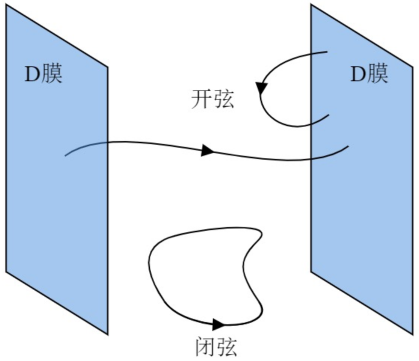
Defects

infinitely heavy
charge probe

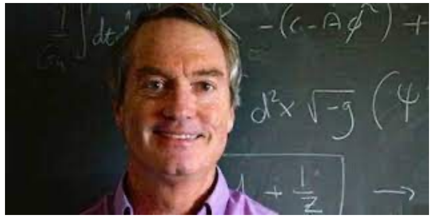


impurity
(Kondo effect)



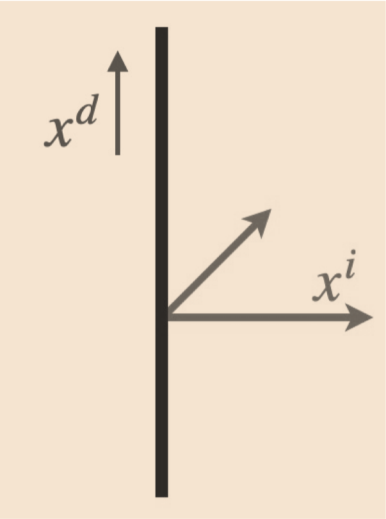


D-brane



Defect in QFT

- Defects are usually defined by non-local operators/B.C. in QFT .



- Defects are classified by their dimensions: line defects, surface defects..(Wilson line, Wilson surface).

- Physically, defect may arise from p-dimensional degrees of freedom coupled to D-dimensional bulk QFT

$$I_{\mathrm{DCFT}} = \int \mathrm{d}^D x \sqrt{g} \mathcal{L}_{\mathrm{CFT}}[\phi] + \int \mathrm{d}^p \hat{x} \sqrt{\gamma} \mathcal{L}_{\mathrm{defect}}[\phi, \psi]$$

Defect landscape

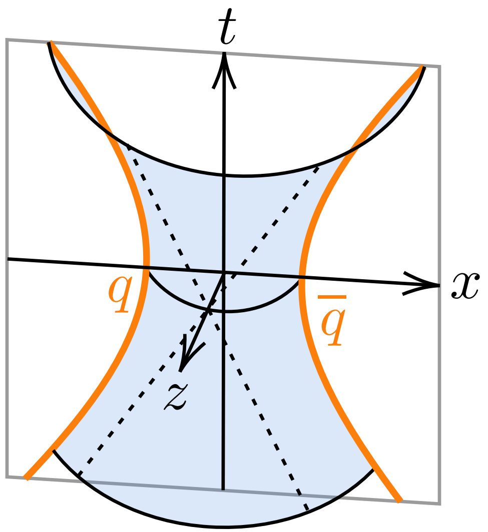
- Topological defects, generalized symmetry.

- Conformal defects, impurities&critical system.

- Boundary, Cross-Cap, monodromy, entangling surface…

Fun with defects

- Bootstrap with defects
- Localization with defects
- Defect RG, Defect ER=EPR
- Defect fusion, intersection [Yifan's talk & Tom's talk]



Conformal defect Kapustin 2006, Billò–Gonçalves–Lauria–Meineri 2016

- Defect breaks part of conformal symmetry.

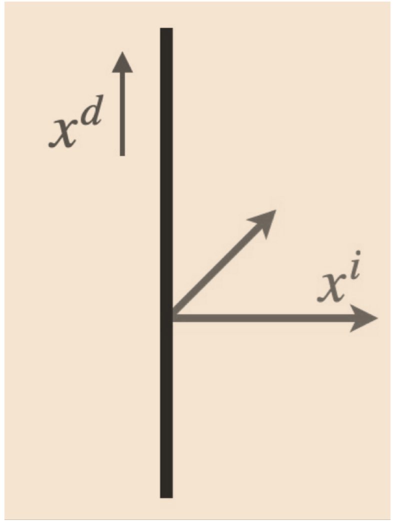
$$SO(1,D+1) \text{ to } SO(1,p+1)\times SO(D-p)$$

- Bulk stress tensor 1-pt function can be fixed up to a constant

$$\langle T^{ab}\rangle = -\frac{D-p-1}{D}\frac{h}{|x^i|^D}\delta^{ab}\ , \quad \langle T^{ai}\rangle = 0\ ,$$

$$\langle T^{ij}\rangle = \frac{h}{|x^i|^D}\left(\frac{p+1}{D}\delta^{ij} - \frac{x^i x^j}{|x^i|^2}\right)\ ,$$

- Displacement operator



Weyl anomaly on surface defect

- For a surface defect, there is an anomaly, analogy to conformal anomaly in CFT2, but now we have b, d_1,d_2

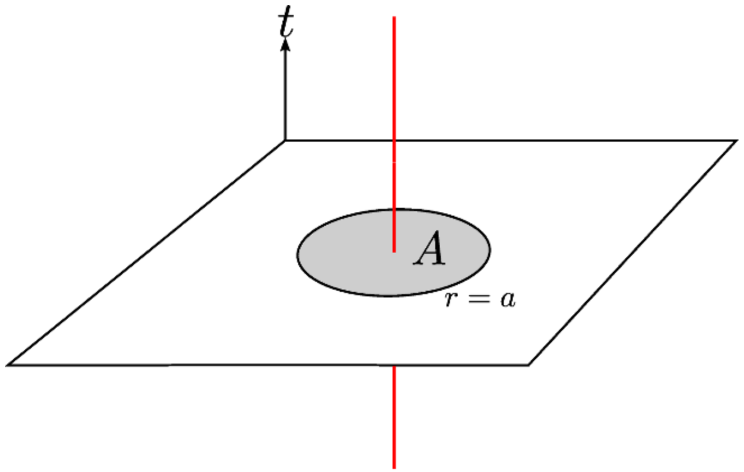
$$\langle\langle \hat{T}^a_{\ a} \rangle\rangle = -\frac{1}{24\pi}\left[bR^{\Sigma} + d_1 \tilde{\Pi}^{\mu}_{ab}\tilde{\Pi}^{ab}_{\mu} - d_2 W^{ab}_{ab}\right]$$

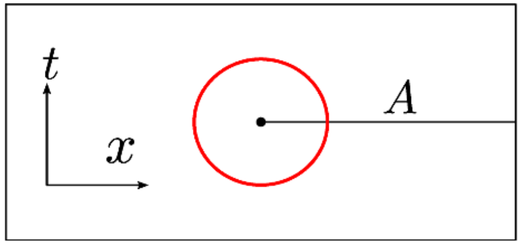
- This can also be seen from log divergence of expectation value

$$\log\langle D_\Sigma\rangle \supset \int_\Sigma \mathcal{A}_\Sigma \mathrm{vol}_\Sigma \log \ell/\epsilon$$

$$\mathcal{A}_\Sigma = \frac{1}{24\pi}\left[bR^{\Sigma} + d_1 \tilde{\Pi}^{\mu}_{ab}\tilde{\Pi}^{ab}_{\mu} - d_2 W^{ab}_{ab}\right]$$

Extra entanglement due to defect Jensen-O'bannon 2013, Lewkowycz-Maldacena 2013





$$S_W = (1 - n\partial_n) \log\langle W \rangle|_{n=1} = \log\langle W \rangle + \int \langle T_{\tau\tau} \rangle_W$$

Interesting observable related to both log<W> and h

Free fields

- free scalar and 2-form in 6d

$$L_\phi = \frac{1}{2}(\partial\phi)^2 + \frac{1}{10}R\phi^2$$

$$L_B = \frac{1}{12}F_{\mu_1\mu_2\mu_3}F^{\mu_1\mu_2\mu_3}$$

- Surface defects in free fields

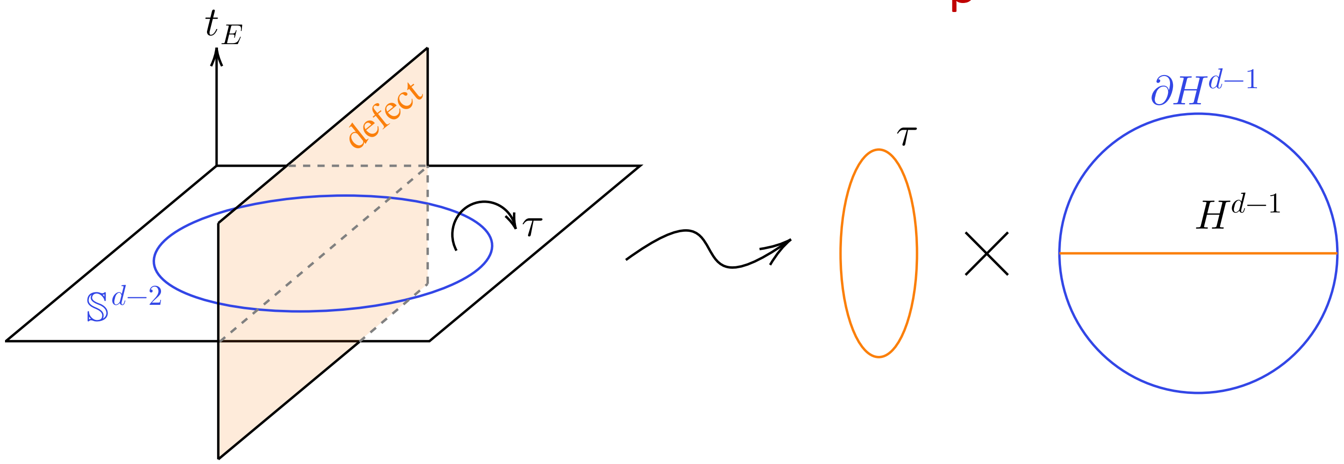
$$D_\phi = \exp\left(\int_\Sigma \mathrm{d}^2\sigma\,\phi(\sigma)\right)$$

$$D_B = \exp\left(i\int_\Sigma B\right) \ .$$

$$h_\phi = \frac{1}{20\pi^4} \ , \quad h_B = \frac{1}{4\pi^4}$$

Renyi entropy

- Map to $S_\beta * H^5$, with defect wrapped on $S_\beta * H^1$



$$ds^2_{\mathbb{R}^d} = dt_E^2 + dr^2 + r^2 d\Omega^2_{d-2} \qquad ds^2_{\mathbb{R}^d} = \Omega^2 ds^2_{S^1 \times H^5} = \Omega^2 \ell^2 (d\tau^2 + d\rho^2 + \sinh^2 \rho \, d\Omega^2_{d-2})$$

$$t_E = \ell \frac{\sin \tau}{\cosh \rho + \cos \tau} \ , \quad r = \ell \frac{\sinh \rho}{\cosh \rho + \cos \tau}$$

Free energy

- Free scalar Kernel on $S_\beta * H^5$

$$K_{H^5}(t,\rho) = \frac{e^{-4t-\frac{\rho^2}{4t}}\left(\rho^2 + 2\rho\, t\coth\rho - 2t\right)\operatorname{csch}^2\rho}{32\pi^{5/2}t^{5/2}}$$

$$K_{S^1_\beta}(t,\tau) = \sum_{n=-\infty}^{+\infty} \frac{\exp\left(-\frac{(\beta n+\tau)^2}{4t}\right)}{\sqrt{4\pi t}}$$

- Integrate propagator over $S_\beta * H^1$

$$G_\beta(\tau,\rho) = \int_0^{+\infty} \mathrm{d}t\, K_{S^1_\beta}(t,\tau) * K_{H^5}(t,\rho) e^{4t}$$

Boundary term

- Additional term on the boundary of H^5, from conformal coupling of the scalar to curvature

$$S = -\frac{4\pi}{10}\int_{\hat{\Sigma}} \mathrm{d}A \langle\langle \phi^2 \rangle\rangle$$

- Evaluate it in the presence of defect,

$$S_\phi = -\frac{1}{10\pi} \log \ell/\epsilon$$

Defect EE

[Kobayashi-Nishioka-Watanabe,2018]
[Jensen-O'Bannon-Robinson-Rodgers 2018]

- Defect EE in terms of central charges was derived before

$$S = \frac{1}{3}\left(b - \frac{d-3}{d-1}d_2\right)\log \ell/\epsilon$$

- Consistency check:

$$S_\phi = -\frac{1}{10\pi}\log \ell/\epsilon \qquad d_2 = 6\pi\Omega_{d-3}\frac{d-1}{d}h_\phi \qquad b = 0$$

Two form

- We use defect EE formula

$$b = \frac{3}{2\pi} \qquad h_B = 1/4\pi^4 \qquad S_B = \frac{1}{3}\left(b - 6\pi^3 h_B\right) \log \ell/\epsilon = 0$$

- From information theory, we conjecture that Renyi entropy also vanishes.

(2,0) free theory

- free tensor multiplet: 5 real scalars + 2 Weyl fermions + 1 two-form(with self-dual strength)

$$W = \exp \int_\Sigma (iB^+ - n^i \phi_i \mathrm{vol}_\Sigma)$$

- Defect contribution to Renyi entropy is a sum,

$$S = S_\phi + \frac{1}{2} S_B = -\frac{1}{10\pi} \log \ell/\epsilon$$

(2,0) in the large N

- CFT6/AdS7 [Maldacena 1997]

$$A_{N-1}\ (2,0)\ \mathrm{SCFT} \qquad\qquad \text{M-theory on } \mathrm{AdS}_7 \times S^4$$

$$\mathrm{d}s_{11}^2 = L^2(\mathrm{d}s^2_{\mathrm{AdS}_7} + \frac{1}{4}\mathrm{d}s^2_{S^4})\ , \quad F_4 = \mathrm{d}C_3 = \pi^2 L^3 \mathrm{vol}_{S^4}\ , \quad L^3 = 8\pi N \ell_p^3$$

- Holographic dual of surface operator in fundamental rep. is a single M2 brane,

$$S_{\mathrm{M2}} = -\log\langle W\rangle$$

Super-Renyi entropy of 6d SCFT

[Nishioka-Yaakov 2013]
[YZ 2016,Yankielowicz-YZ 2017]

- Twist the theory on $S_\beta * H^5$ by R-symmetry leads to supersymmetric Renyi entropy (SRE): a polynomial of $\gamma := 1/n$

$$S_\nu^{(1,0)} = \sum_{n=0}^{3} s_n (\gamma - 1)^n ,$$

$$\begin{aligned} s_0 &= \frac{1}{6}(8\alpha - 8\beta + 8\gamma + 3\delta) , \\ s_1 &= \frac{1}{4}(2\alpha - 3\beta + 4\gamma + \delta) , \\ s_2 &= \frac{1}{24}(2\alpha - 5\beta + 8\gamma) , \\ s_3 &= \frac{1}{192}(\alpha - 4\beta + 16\gamma) \ . \end{aligned}$$

$$\mathcal{I}_8 = \frac{1}{4!}\left(\alpha\, c_2^2(R) + \beta\, c_2(R) p_1(T) + \gamma\, p_1^2(T) + \delta\, p_2(T)\right)$$

[Ohmori-Shimizu-Tachikawa-Yonekura 2014]

Large N limit

- Large N limit is charged hyperbolic black hole with k=-1,m=0

$$\mathrm{d}s_7^2 = -(H_1H_2)^{-4/5} f\mathrm{d}t^2 + (H_1H_2)^{1/5}\left(f^{-1}\mathrm{d}r^2 + r^2\mathrm{d}\Omega_{5,k}^2\right)$$

$$f(r) = k - \frac{m}{r^4} + \frac{r^2}{L^2}H_1H_2\,, \qquad H_i = 1 + \frac{q_i}{r^4}\,,$$

[Cvetic-Duff-Hoxha-Liu-Lu-Lu-Martinez-Acosta-Pope-Sati-Tran 1999]

- One can check holographic SRE

$$\frac{S_\gamma[A_{N\to\infty}]}{N^3} = \frac{1}{192}(\gamma-1)^3 + \frac{1}{12}(\gamma-1)^2 + \frac{1}{2}(\gamma-1) + \frac{4}{3}$$

Defect Super-Renyi

- M2 brane action

$$S_{\rm M2} = T_2 \int \mathrm{d}^3\sigma \sqrt{-\det[g]}$$

- Embedding

$$\sigma_0 = \tau \ , \ \sigma_1 = \rho \ , \ \sigma_2 = r$$

- On-shell action

$$S_{\rm M2} = T_2 \int_0^\beta \mathrm{d}\tau \int_{-\infty}^{\infty} \mathrm{d}\rho \int_{r_H}^{\Lambda} r_H (H_1 H_2)^{-1/5} \sqrt{\Delta}$$

- Large N defect SRE

$$S_\gamma[A_{N\to\infty}] = N\big(r_1 r_2(\gamma - 1) + 2\big) \log \ell/\epsilon$$

$$S_n = \pi T_2 V_{H^1} \left(\frac{7n+1}{16n}\right)$$

A closed formula

[arXiv:2501.09498 with Huang and Yuan]

$$S_\gamma[\mathfrak{g}] = \frac{2b_{\mathfrak{g}} - d_{2\mathfrak{g}}}{6}\left[\frac{r_1 r_2}{2}(\gamma - 1) + 1\right]\log \ell/\epsilon$$

$$r_1 + r_2 = 1$$

- Large gamma defect SRE shares the same behavior with defect SCE(supersymmetric Casimir energy on $S^5 * H^1$)
- Defect SCE is determined from localization, further confirmed by anomaly polynomial
- The constant term is fixed by assuming a linear shift of b/3 by d2

Supersymmetric Casimir energy

[Bobev-Bullimore-Kim 2015]

- 5d partition function from localization (large β)

$$\int \prod_{i=1}^{N} \mathrm{d}\nu_i \exp\left[\frac{2\pi}{\omega_1\omega_2\omega_3}\Big(-\frac{\pi}{\beta}\sum_{i=1}^{N}\nu_i^2+\frac{\sigma_1\sigma_2}{2}\sum_{j<i}(\nu_i-\nu_j)\Big)\right]$$

- Saddle point solution

$$-\frac{2\pi}{\beta}\nu_i+\frac{\sigma_1\sigma_2}{2}\big((i-1)-(N-i)\big)=0 \qquad \Rightarrow \nu_i=\frac{\beta\sigma_1\sigma_2}{4\pi}(2i-N-1)$$

- 6d supersymmetric Casimir energy of A-type (2,0)

$$\beta E_c=-\log\mathcal{Z}=-\beta\frac{N(N^2-1)\sigma_1^2\sigma_2^2}{24\omega_1\omega_2\omega_3}$$

Defect SCE

[arXiv:2501.09498 with Huang and Yuan]
[Mori-Yamaguchi 2014]

- Wilson loop insertion in localization $\mathrm{Tr}_{\mathcal{R}} \exp\left[2\pi\nu/\omega_j\right]$

$$\langle W_{(k)}\rangle = \frac{1}{\mathcal{Z}} \int \prod_{i=1}^{N} \mathrm{d}\nu_i \exp\left[\frac{2\pi}{\omega_1\omega_2\omega_3}\left(-\frac{\pi}{\beta}\sum_{i=1}^{N}\nu_i^2 + \frac{\sigma_1\sigma_2}{2}\sum_{j<i}(\nu_i-\nu_j) + \omega_2\omega_3 k\nu_N\right)\right]$$

- For symmetric representation

$$\begin{aligned}\beta E_{(k)} &= -\log\langle W_{(k)}\rangle \\ &= -\frac{\beta}{2\omega_1}\left(k(N-1)\sigma_1\sigma_2 + k^2\omega_2\omega_3\right)\end{aligned}$$

- For anti-symmetric rep.

$$\beta E_{[k]} = -\frac{\beta}{2\omega_1}\left(k(N-k)\sigma_1\sigma_2 + k\omega_2\omega_3\right)$$

Defect SCE from anomaly polynomial

- Anomaly polynomial from inflow [Shimizu-Tachikawa 2016, Wang 2021]

$$I_4 = \frac{1}{4}(\Lambda,\Lambda)\left(c_2(F_L) - c_2(F_R)\right) + \frac{1}{2}(\Lambda,\rho)\left(c_2(F_I) - c_2(F_F)\right)$$

- Equivariant integration

$$E_{\mathfrak{g}} = -\frac{1}{2\pi}\int I_4 = -\frac{1}{4\omega_1}\left[\frac{1}{2}(\Lambda,\Lambda)\left(a_L^2 - a_R^2\right) + (\Lambda,\rho)\left(a_I^2 - a_F^2\right)\right]$$

- Identify 2d and 6d parameters and use the recent defect anomaly formula

$$b = 24\,(\Lambda,\rho) + 3\,(\Lambda,\Lambda)\ ,\quad d_2 = 24\,(\Lambda,\rho) + 6\,(\Lambda,\Lambda)$$

$$E_{\mathfrak{g}} = -\frac{1}{\omega_1}\left[\frac{d_{2\mathfrak{g}} - b_{\mathfrak{g}}}{6}\omega_2\omega_3 + \frac{2b_{\mathfrak{g}} - d_{2\mathfrak{g}}}{24}\sigma_1\sigma_2\right]$$

$$\omega_1 = \omega_2 = 1,\ \sigma_1\sigma_2 = 2\omega_3$$

$$E_{\mathfrak{g}}\big|_{\text{chiral limit}} = -\frac{1}{12}\omega_3 d_2$$

[Chalabi-O'bannon-Robinson-Sisti 2020]

Conclusion&Discussion

- We propose a closed formula for defect contribution to Super-Renyi entropy as well as supersymmetric Casimir energy in terms of central charges in M5 brane theories.

- A defect Cardy formula?

- Bounds on defect anomalies?

- (1,0)? Other dimensional defects? Gukov-Witten in 4d?