Extreme Universe

Localized RG flows on composite defects & C-theorem

Dongsheng Ge

(Osaka University)

"Non-perturbative methods in QFT"

13.03.2025 @ Kyushu U. IAS

OSAKA UNIVERSITY

In collaboration with

Tatsuma Nishioka

Soichiro Shimamori

arXiv: 2408.04428

Defects are interesting objects in physics

[Yifan's talk] [Ingo's talk]
[Naoto's talk] [Soichiro's talk]
[Yang's talk]

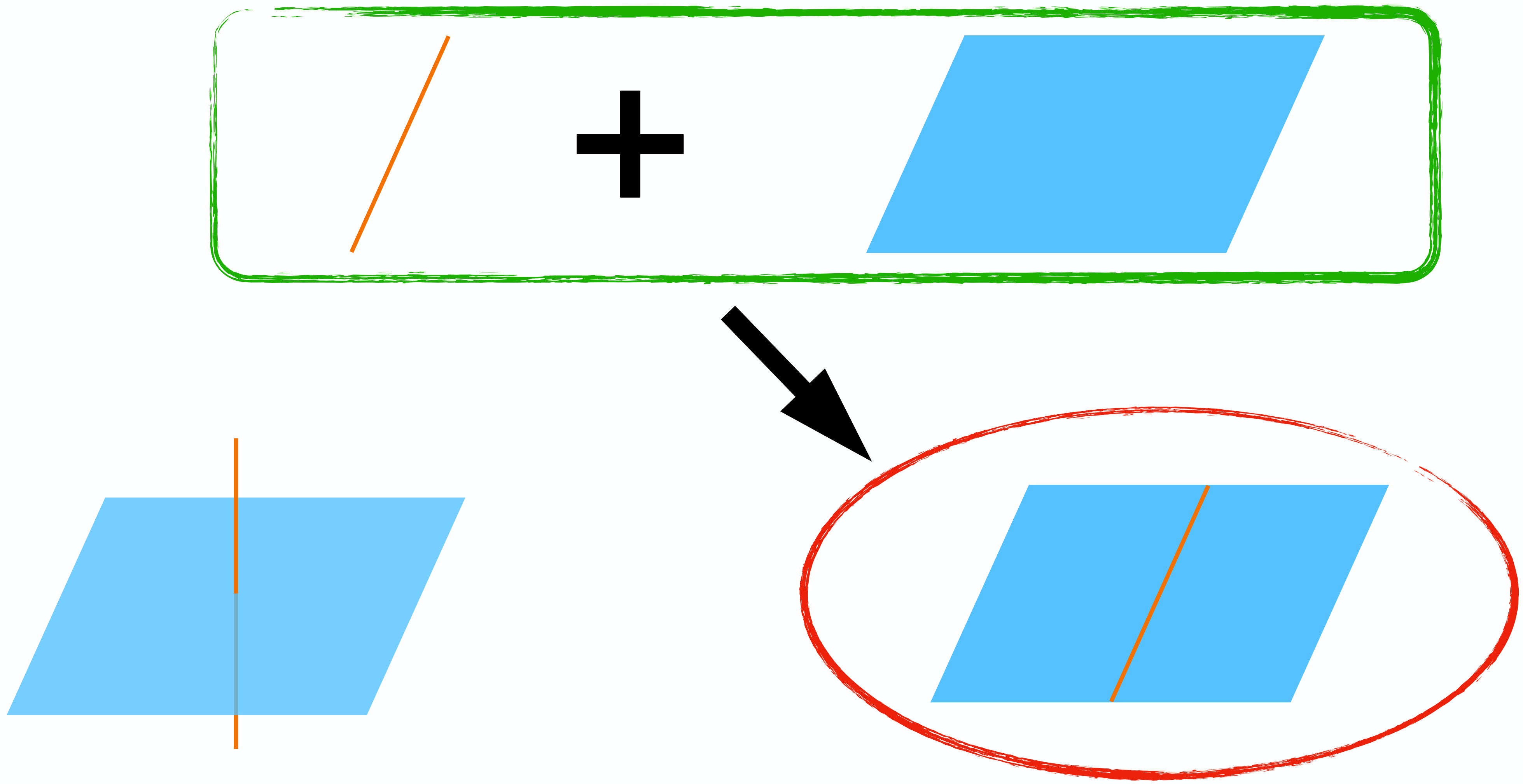
- Line defect: Wilson-'t Hooft line, impurities, cosmic strings, ...
- Higher-dimensional defects: D-branes, ...
- Topological defects: Skyrmions (2D), Hopfions (3D), ...

Normally, only one defect is considered in a physical system. What if there exists more than one defect ? In this situation, perhaps we can do more engineering. The simplest situation is to consider two defects.

- Two of the same dimension
- Two of different dimensions (spacetime $D > 2$)

What is a composite defect ?

We can start by doing some engineering, taking a line defect and a surface defect



When a lower-dimensional defect is embedded inside a higher-dimensional defect, this gives rise to a composite defect.

My plans:

- Defects as localized interactions

- What does not work: Free O(N) model in $d = 4 - \varepsilon$

- What does work: Free O(N) model $d = 3 - \varepsilon$

- A (weak) C-theorem conjecture for the composite defects

- Summary and open questions

Defects as Localized interactions

Defects as Localized interactions

In the model we are considering, defects are represented by the localized interactions. This means that they can flow under local renormalization group (RG) flow. The local deformations are marginal classically. For example, a scalar field has a classical dimension

$$\Delta_\phi = \frac{d}{2} - 1$$

We can build local deformations of the type $\int d^r x \phi^q$

Being classically marginal requires that $q\Delta_\phi = r$

d	r	q
4	1	1
4	2	2
3	1	2
3	2	4
...	...	...

To suit our purposes, we would like to construct TWO such defects of dimensions r and p (r < p).

$$S_{\mathrm{d}_1} = \frac{g_0}{l!}\int_{\mathbb{R}^r} d^r y \phi^l \,, \quad S_{\mathrm{d}_2} = \frac{h_0}{k!}\int_{\mathbb{R}^p} d^p z \phi^k \,,$$

$$r,\ k,\ p,\ l \in \mathbb{Z}^+,\ 1 \le r < p < d \text{ and } l < k$$

Requiring them being classically marginal, we have the following allowed configurations (one should think d= m - ε , ε -> 0)

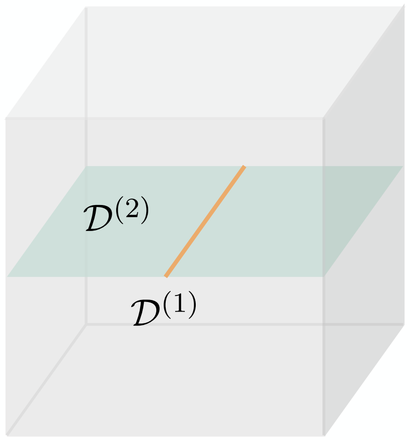
d	3	4			$2n$
r	1	1	1	2	$n-1$
l	2	1	1	2	1
p	2	2	3	3	$2n-2$
k	4	2	3	3	2

$$d = 2n - \varepsilon \quad (n = 2)$$

A single scalar model with composite defect

Model: Line defect + surface defect

$$I = \frac{1}{2}\int d^dx\,(\partial\phi)^2 + \frac{h_0}{2}\int_{\mathbb{R}^2} d^2\hat{y}\,\phi^2 + g_0\int d\tilde{z}\,\phi$$



Epsilon expansion of the bare couplings,

$$h_0 = M^{\epsilon}\left(h + \frac{\delta h}{\epsilon} + \frac{\delta_2 h}{\epsilon^2} + \dots\right),$$
$$g_0 = M^{\epsilon/2}\left(h + \frac{\delta g}{\epsilon} + \frac{\delta_2 g}{\epsilon^2} + \dots\right)$$

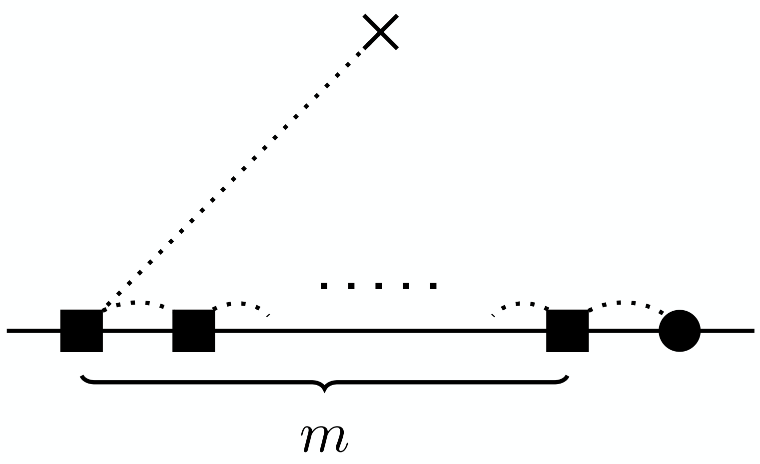
Regularization of the theory, one-point function of the renormalized field should be finite

$$\langle[\phi](x)\rangle = \text{finite}\,, \quad \langle[\phi^2](x)\rangle = \text{finite}\,.$$

All loop calculation

Type I diagram: Chain type $\langle \phi(x) \rangle$

$\Gamma_m(x) \;=$



■ Surface defect

● Line defect

We can use the free field propagator to calculate perturbatively

$$\Gamma_m(x) = (-g_0)\frac{(-h_0/2)^m}{m!}\int \mathrm{d}^r\tilde{z}\int\prod_{i=0}^{m}\mathrm{d}^{2r}\hat{y}_i\,\langle\,\phi(x)\prod_{j=0}^{m}\hat{\phi}^2(\hat{y}_i)\,\tilde{\phi}(\tilde{z})\,\rangle$$

$$= -g_0(-h_0)^m\int\frac{\mathrm{d}^{d-r}\tilde{p}}{(2\pi)^{d-r}}\boxed{f_r^m(|\tilde{p}_{\parallel,H}|)}\frac{e^{-\mathrm{i}\,\tilde{p}x}}{\tilde{p}^2}$$

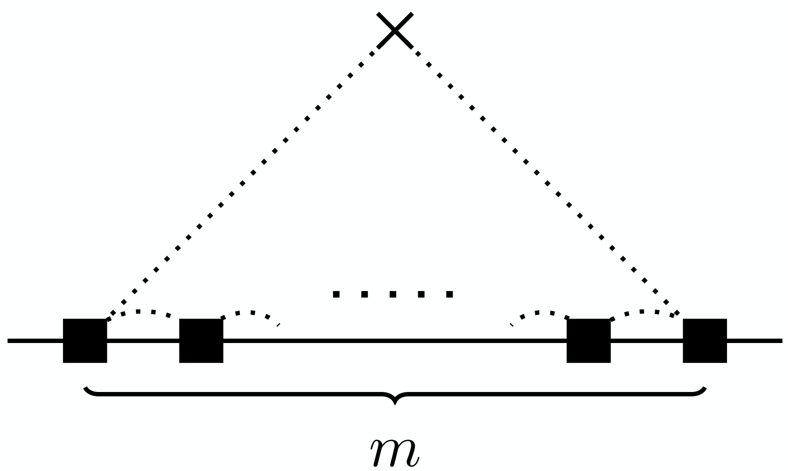
$$\boxed{f_r(|\tilde{p}_{\parallel,H}|) = (4\pi)^{\frac{\epsilon-2}{2}}\,|\tilde{p}_{\parallel,H}|^{-\epsilon}\,\Gamma\left(\frac{\epsilon}{2}\right)}$$

Summing up all the diagrams gives

$$\langle\phi(x)\rangle = (-g_0)\int\frac{\mathrm{d}^{d-r}\tilde{p}}{(2\pi)^{d-r}}\,\frac{1}{1+h_0 f(|\tilde{p}_{\parallel,H}|)}\,\frac{e^{-\mathrm{i}\,\tilde{p}x}}{\tilde{p}^2}$$

Type II diagram: Loop type, with only surface defects $\langle\phi^2(x)\rangle$

$\tilde{\Gamma}_m(x) \ =$



$$\tilde{\Gamma}_m(x) = \frac{(-h_0/2)^m}{m!}\int \prod_{i=1}^{m} \mathrm{d}^{2r}\hat{y}_i \langle \phi^2(x) \prod_{j=1}^{m} \hat{\phi}^2(\hat{y}_j)\rangle$$
$$= (-h_0)^m \int \frac{\mathrm{d}^{d-2r}\hat{p}}{(2\pi)^{d-2r}} \frac{\mathrm{d}^d p}{(2\pi)^d} \frac{e^{-ix\hat{p}}}{(\hat{p}+p)^2 p^2} f_r^{m-1}(|p_{\|,H}|)$$

Summing up all the diagrams gives the non-factorizable part

$$\langle\phi^2(x)\rangle = (\langle\phi(x)\rangle)^2 + \int \frac{\mathrm{d}^{d-2r}\hat{p}}{(2\pi)^{d-2r}} \frac{\mathrm{d}^d p}{(2\pi)^d} \frac{e^{-ix\hat{p}}}{(\hat{p}+p)^2 p^2} \frac{-h_0}{1+h_0 f_r(|p_{\|,H}|)}$$

Finiteness conditions give the renormalized coupling

$$\frac{M^{\epsilon/2}}{g_0} + \frac{h_0}{g_0}\frac{M^{-\epsilon/2}}{2\pi\epsilon} = \frac{1}{g}\,, \qquad \frac{M^{\epsilon}}{h_0} + \frac{1}{2\pi\epsilon} = \frac{1}{h}$$

$$\Downarrow$$

$$\beta_h = \frac{\mathrm{d}h}{\mathrm{d}\ln M} = \frac{h(h-2\pi\epsilon)}{2\pi}\,, \qquad \beta_g = \frac{\mathrm{d}g}{\mathrm{d}\ln M} = \frac{g(h-\pi\epsilon)}{2\pi}$$

<u>Comments</u>:

- If "2" is not there…
- The line coupling is corrected by the surface coupling, but not vice versa
- The localized RG flows on the surface and the line cannot flow to the fixed point simultaneously -> no conformal composite defect
- This analysis applies for free theories in even dimensions (n>2), they cannot host conformal composite defects

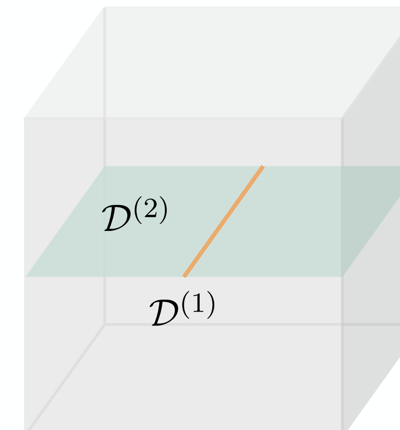
$$d = 3 - \varepsilon$$

O(N) model with composite defects

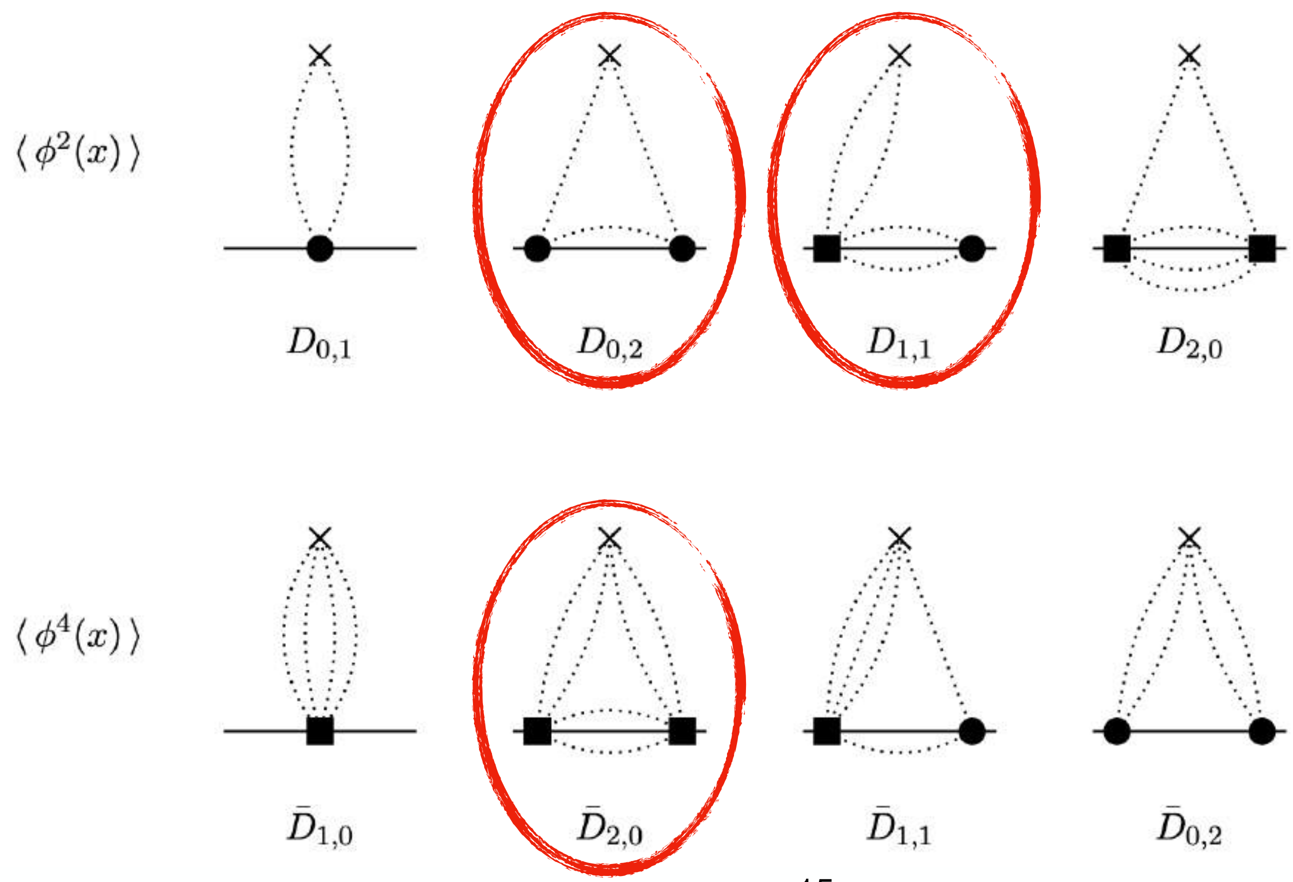
[Tom Shachar's talk]

Model: surface defect + line defect

$$I \equiv \frac{1}{2}\int_{\mathbb{R}^d} \mathrm{d}^d x\, \partial_\mu \phi^I \partial^\mu \phi^I + \frac{h_0}{4!}\int_{\mathbb{R}^2} \mathrm{d}^2 \hat{y}\, \left(\phi^I\phi^I\right)^2 + \frac{g_{0,IJ}}{2}\int_{\mathbb{R}} \mathrm{d}\tilde{z}\, \phi^I \phi^J$$



Diagrams up to the second order

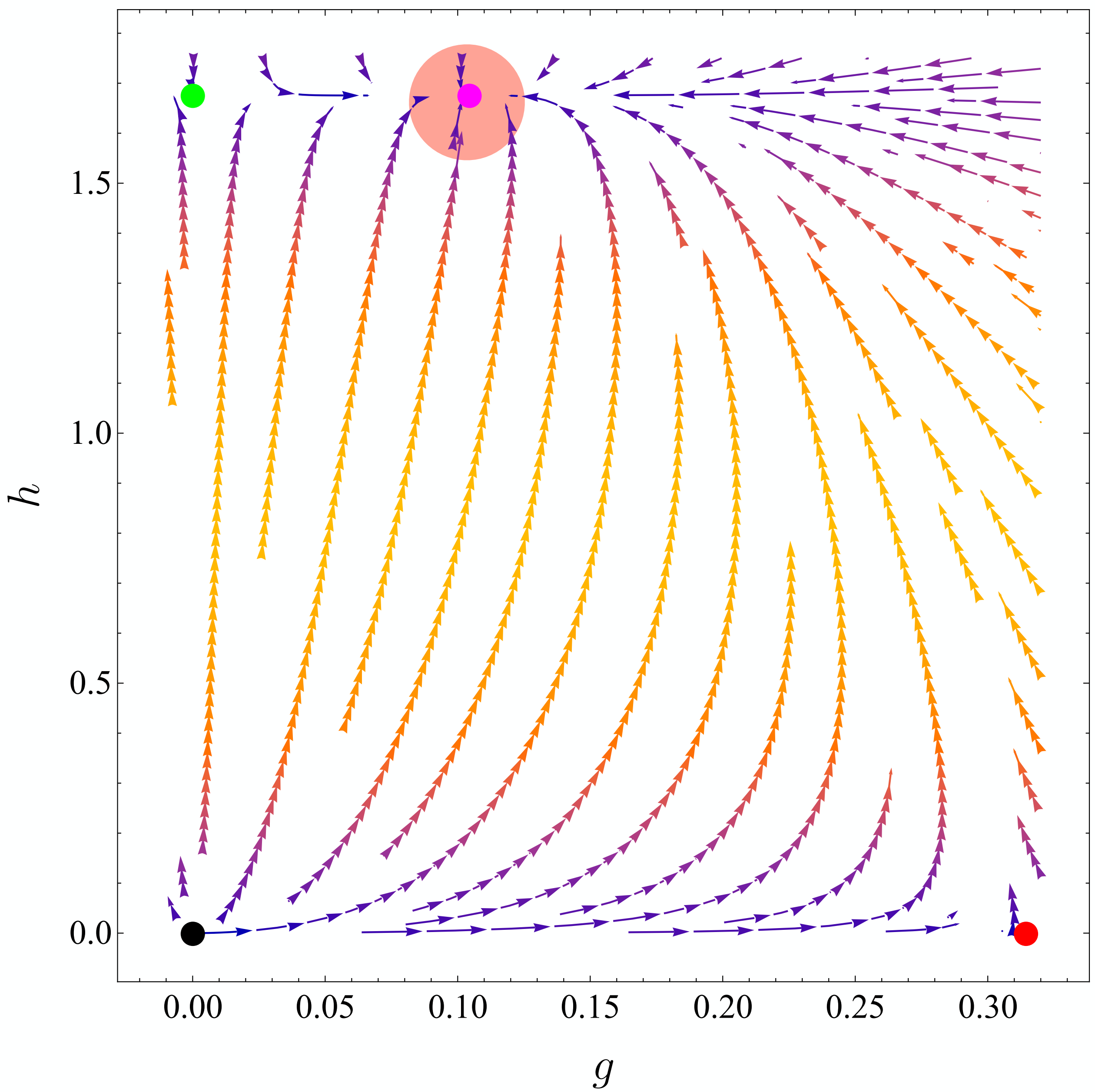


■ Surface defect

● Line defect

Divergent diagram

O(1) model: only a single scalar



- UV
- Line defect
- Surface defect
- Composite defect

Comments:

- In this dimension, both conformal line and surface defects can be hosted separately
- An exciting fact: conformal composite defect exists !!!
- At the CCD fixed point, line coupling is modified by the surface one

$$\beta_h = -2\epsilon h + \frac{3h^2}{16\pi}$$

$$\beta_g = -\epsilon g + \frac{g^2}{2\pi} + \frac{hg}{16\pi}$$

$$h_* = \frac{32\pi}{3}\epsilon + O(\epsilon^2)\,,$$

$$g_* = 2\pi\epsilon - \frac{h_*}{8} + O(\epsilon^2) = \frac{2\pi}{3}\epsilon + O(\epsilon^2)$$

More generally O(N) case: O(N) -> O(m) * O(N-m) on the line

$$g_{IJ} = \mathrm{diag}\,(\underbrace{g\,,\,g\,,\cdots,\,g}_{m}\,,\,\underbrace{g'\,,\,g'\,,\cdots,\,g'}_{N-m})$$

Beta functions:

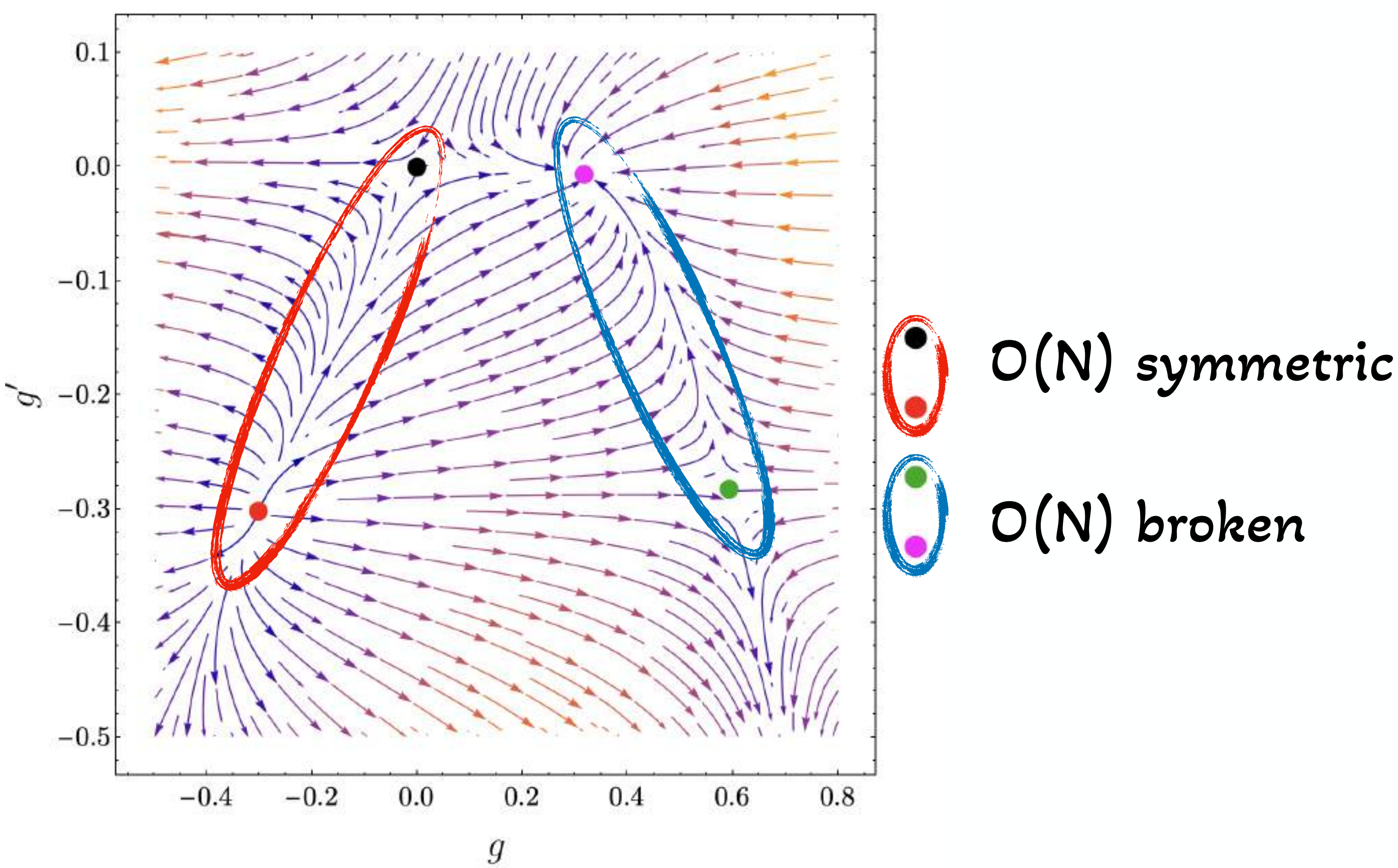
$$\beta_h = -2\epsilon\,h + \frac{N+8}{48\pi}\,h^2 + (\text{higher order})$$

$$\beta_g = -\epsilon\,g + \frac{1}{2\pi}g^2 + \frac{1}{48\pi}(m\,g + (N-m)g' + 2g)h + \cdots\,,$$

$$\beta_{g'} = -\epsilon\,g' + \frac{1}{2\pi}g'^2 + \frac{1}{48\pi}(m\,g + (N-m)g' + 2g')h + \cdots\,.$$

RG flows plotted w/ fixed surface coupling

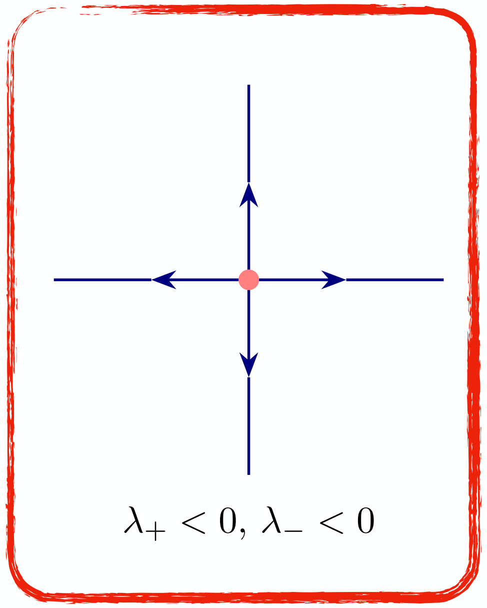
$$h_* = \frac{96\,\pi}{N+8}\,\epsilon + O(\epsilon^2)$$



Comments:

- Four CCD fixed points, 2 * O(N) symmetric + 2 * O(N) broken
- Unitarity (real couplings) requires N >= 23
- Local RG analysis shows that the non-trivial O(N) preserving one is the most UV among the four for N > 4. We can perturb the coupling around the fixed point, obtaining the eigenvalues for the phase space (g,g')

$$\lambda_+ = -\frac{N-4}{N+8}\epsilon\,, \qquad \lambda_- = -\frac{3N-4}{N+8}\epsilon$$



dCFT data: An interesting mixing of composite operators

We consider two composite operators of the same classical dimension on the line defect

$$\Phi \equiv \frac{1}{\sqrt{m}} \sum_{\alpha=1}^{m} (\phi^\alpha)^2 \, , \qquad \Psi \equiv \frac{1}{\sqrt{N-m}} \sum_{i=m+1}^{N} (\phi^i)^2 \, , \qquad \begin{pmatrix} \Phi \\ \Psi \end{pmatrix} = Z_S \begin{pmatrix} [\Phi] \\ [\Psi] \end{pmatrix}$$

Apparently, once the surface coupling is turned on, at the quantum level, the two operators stop to be orthogonal to be each other, as the wavefunction renormalization matrix acquires off-diagonal entries. We can define a matrix for the anomalous dimension

$$\gamma \equiv Z_S^{-1} \frac{\mathrm{d}}{\mathrm{d} \log M} Z_S \Big|_{\text{fixed point}}$$

Its eigenvalues give rise to the anomalous dimensions of two new defect operators, for instance, at the non-trivial O(N) symmetric fixed point,

$$\boxed{\gamma_\pm \Big|_{P_1} = \frac{12 - N \pm N}{8 + N} \epsilon}$$

Conjecture: Subdefect C-theorem

C - theorems

We can define a function, normally called a c-function after Zamolodchikov. This function has monotonic properties along the RG flow. In 2d, it coincides with the central charge s.t. it depicts the d.o.f. of the theory at different energy scales and UV has more d.o.f. than the IR. There are three versions of C-theorem

[Zamolodchikov 1986]

- Weak version: C-function is non-increasing compared at the UV and the IR fixed points connected by an RG flow;
- Strong version: C-function is non-increasing along the RG flow;
- Strongest version: the RG flow is a gradient flow of the C-function.

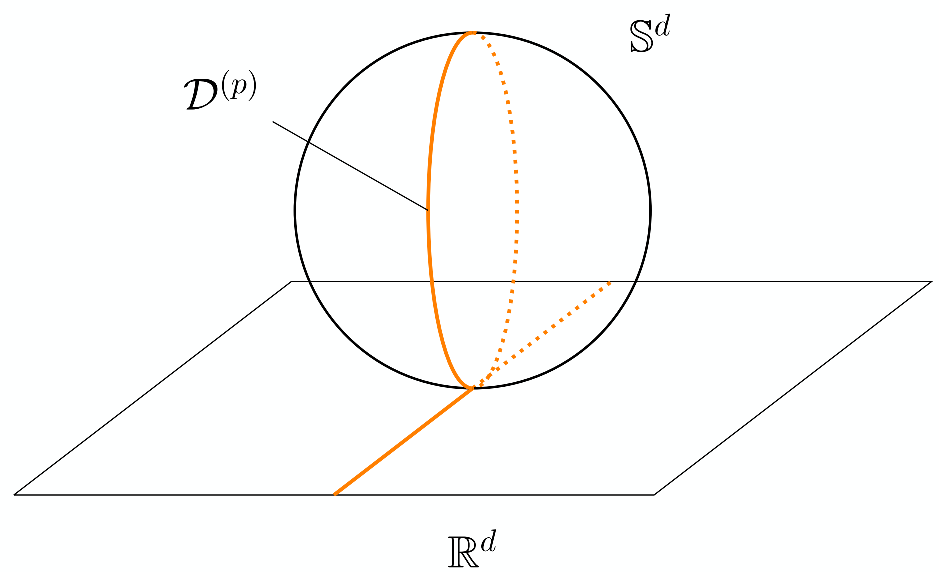
A conjecture of the sub-defect C - theorem

Conjecture. In a unitary CFT$_d$ with a composite defect $\mathcal{D}^{(p_1,\cdots,p_n)} = \cup_{i=1}^n \mathcal{D}^{(p_i)}$ consisting of n sub-defects of p_i dimensions satisfying $0 < p_1 < p_2 < \cdots < p_n < d$ and $\mathcal{D}^{(p_1)} \subset \mathcal{D}^{(p_2)} \subset \cdots \subset \mathcal{D}^{(p_n)}$, let $Z^{(p_1,\cdots,p_n)} \equiv \langle \mathcal{D}^{(p_1,\cdots,p_n)} \rangle$ be the partition function on a d-sphere. Then, the function $\mathcal{C}$ defined by

$$\mathcal{C} \equiv \sin\left(\frac{\pi p_1}{2}\right) \log \left| \frac{Z^{(p_1,p_2,\cdots,p_n)}}{Z^{(p_2,\cdots,p_n)}} \right| \tag{4.1}$$

does not increase under any localized RG flow on the sub-defect $\mathcal{D}^{(p_1)}$ of the lowest dimension,

$$\mathcal{C}_{\mathrm{UV}} \geq \mathcal{C}_{\mathrm{IR}} \ . \tag{4.2}$$



An argument from conformal perturbation theory

Let us consider the following weakly relevant deformation localized on the submost defect,

$$I^{(p_1,\cdots,p_n)} = I^{(p_1,\cdots,p_n)}_{\text{DCFT}} + \tilde{\lambda}_0 \int \mathrm{d}^{p_1}\tilde{x}\,\sqrt{\tilde{g}}\,\widetilde{\mathcal{O}}(\tilde{x}) \ , \qquad \boxed{\widetilde{\Delta} = p_i - \varepsilon}$$

Adopting the Wilsonian renormalization procedure, we can find the renormalized (dimensionless) coupling $\tilde{g}(\mu) = \tilde{\lambda}(\mu)\,\mu^{-\varepsilon}$ and $\tilde{g}_0 = \tilde{\lambda}_0\,\mu_{\text{UV}}^{-\varepsilon}$

$$\tilde{g}(\mu) = \tilde{g}_0\left(\frac{\mu_{\text{UV}}}{\mu}\right)^{\varepsilon} - \tilde{g}_0^2\,\frac{\pi^{\frac{p_1}{2}}\,\widetilde{C}}{\varepsilon\,\Gamma\left(\frac{p_1}{2}\right)}\left[\left(\frac{\mu_{\text{UV}}}{\mu}\right)^{2\varepsilon} - \left(\frac{\mu_{\text{UV}}}{\mu}\right)^{\varepsilon}\right] + \cdots$$

Then the difference of the partition functions up to third order in the coupling

$$\begin{aligned}\delta\log\,Z^{(p_1,\cdots,p_n)}(\tilde{g}) &\equiv \log\,Z^{(p_1,\cdots,p_n)}(\tilde{g}) - \log\,Z^{(p_1,\cdots,p_n)}(0)\\ &= \frac{2\,\pi^{p_1+1}}{\sin\left(\frac{\pi p_1}{2}\right)\,\Gamma(p_1+1)}\left[-\frac{\varepsilon}{2}\,\tilde{g}^2 + \frac{\pi^{\frac{p_1}{2}}\,\tilde{C}}{3\,\Gamma\left(\frac{p_1}{2}\right)}\,\tilde{g}^3 + O(\tilde{g}^4)\right]\end{aligned}$$

This gives rise to the C-function

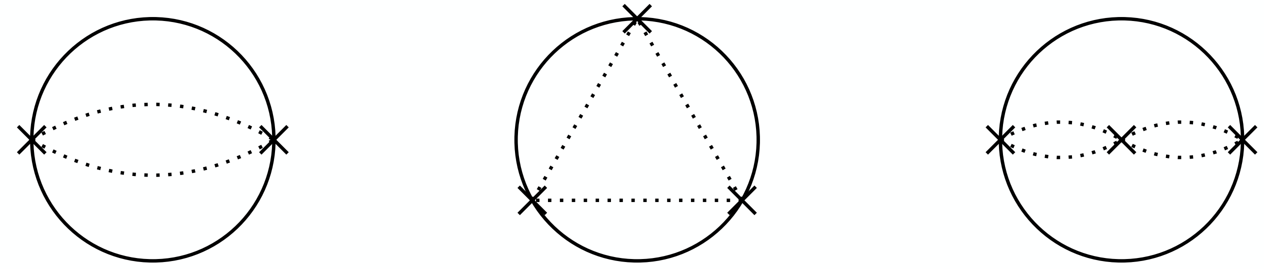
$$\boxed{\mathcal{C}(\tilde{g}_*) - \mathcal{C}(\tilde{g}=0) = -\frac{\pi\,\Gamma\left(\frac{p_1}{2}\right)^2}{3\,\Gamma(p_1+1)}\,\frac{\varepsilon^3}{\widetilde{C}^2} + O(\varepsilon^4)}$$

Perturbative test in the d = 3 - ε model

The conjectured sub-defect C-function for our model consists of

$$\mathcal{C} = \ln \frac{Z(h_0, g_0, g_0')}{Z(h_0, 0, 0)} = C_2 + C_3 + C_3' + \text{(higher order)}$$

Diagrammatically,



C_2 C_3 C_3'

$$\mathcal{C} = -\frac{m\, g^3 + (N-m)\, g'^3}{192\, \pi} + \frac{m\, g\, \beta_g + (N-m)\, g'\, \beta_{g'}}{32}$$

Comments:

- Stationary around the fixed points, $\frac{\partial \mathcal{C}}{\partial g} = -\frac{m}{16}\, \beta_g\,, \qquad \frac{\partial \mathcal{C}}{\partial g'} = -\frac{N-m}{16}\, \beta_{g'}$
- Agrees with Local RG analysis, comparing the two O(N) symmetric fixed points $\mathcal{C}(P_1) - \mathcal{C}(P_0) = \frac{\pi^2\, N\, (N-4)^3}{24\, (N+8)^3} \epsilon^3 + O(\epsilon^4)$

Future directions:

- Physical relevance of the composite defect, $d = 3 - \varepsilon$?

[In preparation w/ Nakayama]

- Adding a bulk interaction and re-investigate the $d = 4 - \varepsilon$, line surface & volume(interface) [WIP w/ Stergiou & Pannell]

- Proof of the subdefect C-theorem ?

An interesting open question:

- How can we fuse defects of different dimensions ?

[Bachas & Brunner '07]

[Diatlyk, Khanchandani, Popov & Wang '24]

ありがとう