Non-Invertible Symmetry and Entanglement Entropy

Based on arXiv 2409.02159 and 2409.02806 (PRL)
with Brandon C. Rayhaun and Yunqin Zheng

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Plan of the Talk

1. Review: Entanglement Entropy and BCFT [Ohmori-Tachikawa ’14; …]

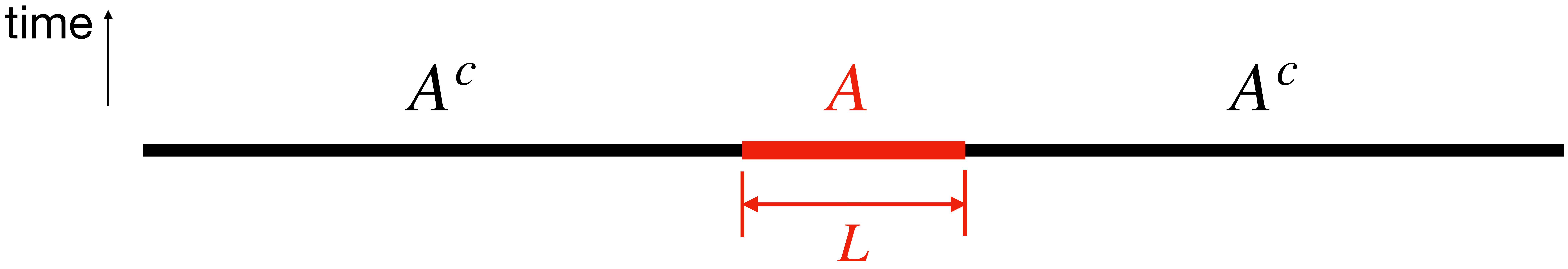
2. Symmetries and Conformal Boundaries in 2D

3. “Topological Holography” for Conformal Boundaries

4. Non-Invertible Symmetries and Entanglement Entropy

Entanglement Entropy and BCFT

Entanglement Entropy in 2D CFT



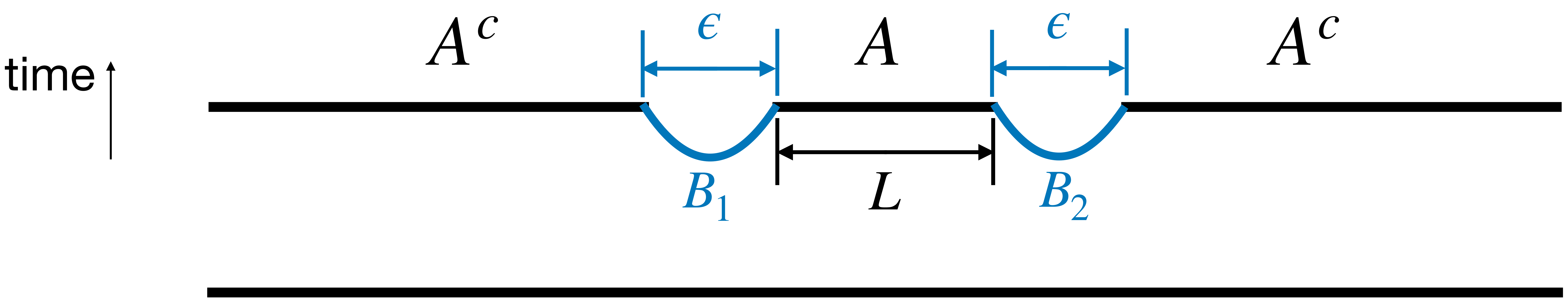
Famously, $S_A = -\,\mathrm{Tr}\left(\rho_A \log \rho_A\right) = \frac{c}{3}\log\frac{L}{\epsilon} + \cdots$ [Holzhey-Larsen-Wilczek '94]

where " $\rho_A = \mathrm{Tr}_{\mathscr{H}_{A^c}} |\Omega\rangle\langle\Omega|$ " is the reduced density matrix of the interval A.

Entanglement Entropy in 2D CFT

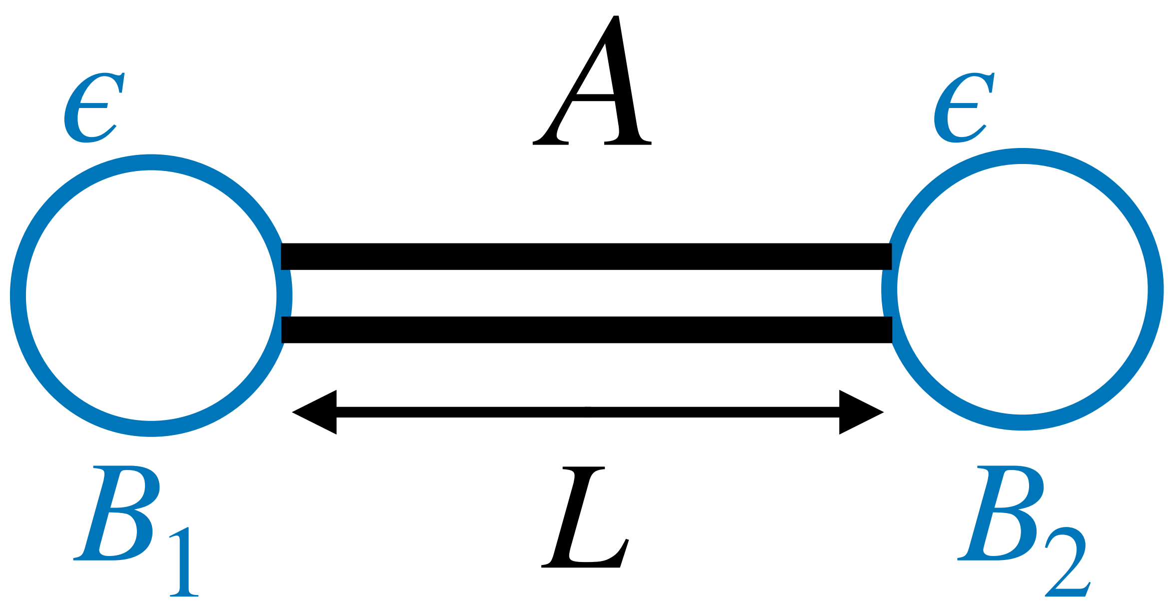
- Strictly speaking, the expression “ $\rho_A = \mathrm{Tr}_{\mathscr{H}_{A^c}} |\Omega\rangle\langle\Omega|$ ” does not make sense in continuum field theories.
- The Hilbert space does not factorize: $\mathscr{H} \neq \mathscr{H}_A \otimes \mathscr{H}_{A^c}$.
- Relatedly, the entanglement entropy naively diverges, and requires regularization.
- We follow [Ohmori-Tachikawa ’14], and will introduce explicit conformal boundary conditions at the entanglement cuts. [Yang Zhou’s talk]

Conformal Boundaries and EE



- To obtain an explicitly factorized Hilbert space, we consider a "topology-changing" Euclidean path integral.
- At the semicircles, we impose conformal boundary conditions B_1 and B_2.
- The path integral defines a map $\iota : \mathcal{H} \to \mathcal{H}_A \otimes \mathcal{H}_{A^c}$.

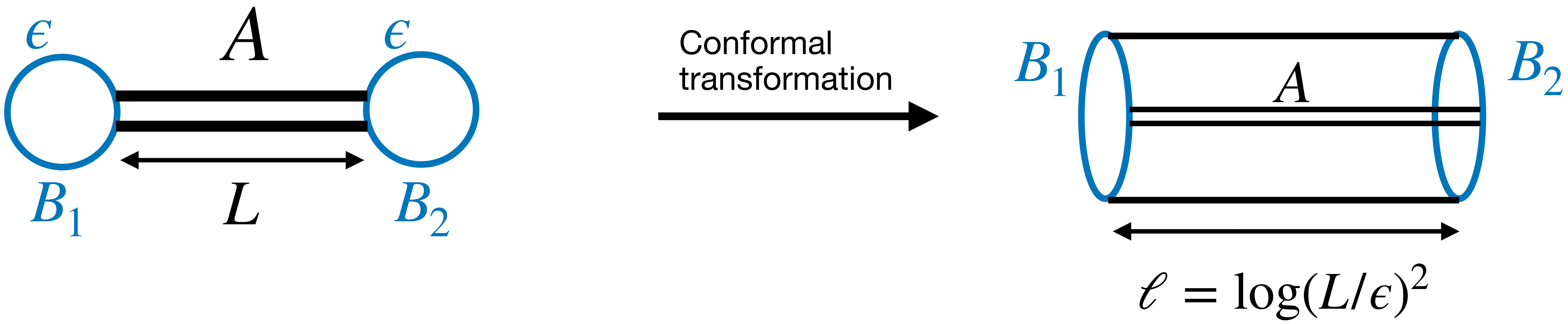
Conformal Boundaries and EE



- We define the reduced density matrix of the single interval A of the vacuum state $|\Omega\rangle$ by $\rho_A \equiv \mathrm{Tr}_{\mathscr{H}_{A^c}} \iota |\Omega\rangle\langle\Omega| \iota^\dagger$.

- ρ_A is obtained by the Euclidean path integral on a plane with two small disks and a slit between them excised.

Conformal Boundaries and EE



- By performing a conformal transformation $z \mapsto \log(z) - \log(L - z)$, we map to a cylinder (with a small slit excised) with circumference 2π and length ℓ.
- The reduced density matrix is determined by the open string Hamiltonian:

$$\rho_A = \frac{e^{-2\pi H_{open}}}{\mathrm{Tr} e^{-2\pi H_{open}}} \quad \text{where } H_{open} = \frac{\pi}{\ell}\left(L_0 - \frac{c}{24}\right).$$

Conformal Boundaries and EE

$$\rho_A = \frac{e^{-2\pi H_{open}}}{\mathrm{Tr} e^{-2\pi H_{open}}}$$

- By using the explicit form of ρ_A, we can compute the entanglement entropy of the interval A, [cf. Nathan Benjamin's talk]

$$S_A = \frac{c}{3} \log \frac{L}{\epsilon} + \log g_1 + \log g_2 + \cdots .$$

- Here, $g_i = \langle 0 | B_i \rangle$ are the g-functions of conformal boundaries B_i.
- $\log g_i$ is also known as the Affleck-Ludwig boundary entropy.

EE = BCFT correspondence

EE	BCFT
Subregion Hilbert space $\mathscr{H}_A$	Open string Hilbert space $\mathscr{H}_{B_1 B_2}$
Reduced density matrix ρ_A	$\dfrac{e^{-2\pi H_{open}}}{\mathrm{Tr} e^{-2\pi H_{open}}}$
Entanglement Hamiltonian	$H_{open} = \dfrac{\pi}{\ell}\left(L_0 - \dfrac{c}{24}\right)$

- Once we choose explicit conformal boundary conditions B_1 and B_2 at the entanglement cuts, the EE problem is mapped to the corresponding BCFT problem.

EE = BCFT correspondence

- What determines the conformal boundary condition at the entanglement cuts?
- From the continuum field theory point of view, the choice is up to us. Different choices give the same universal result at the leading order.
- On the other hand, in many physical situations, we may also be interested in a particular microscopic realization, e.g. condensed matter systems at critical points.
- The correspondence with BCFT has been numerically tested in various critical lattice models [Lauchli '13].
- Comparing the entanglement spectrum with the BCFT spectrum gives more detailed information about the fixed point beyond central charge.

Symmetries of Entanglement Hamiltonian

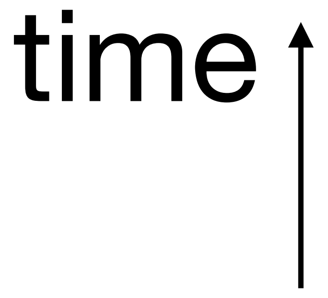
- Suppose the 2D CFT we are interested in has a (finite) non-invertible global symmetry described by a fusion category. [Ingo Runkel’s talk]
- How are these symmetries realized by the Entanglement Hamiltonian? What are their imprints?
- How does the Entanglement Spectrum organize into representations of the global symmetry? How frequently a given representation appear?
- For ordinary symmetries, this was studied in [Goldstein-Sela; Magan; Casini-Huerta-Magan-Pontello; Kusuki-Murciano-Ooguri-Pal;…], and corresponding Symmetry-Resolved Entanglement Entropies were computed. [Higher dimensions: Huang-Zhou (yesterday)]

Symmetries and Boundaries in 2D CFTs

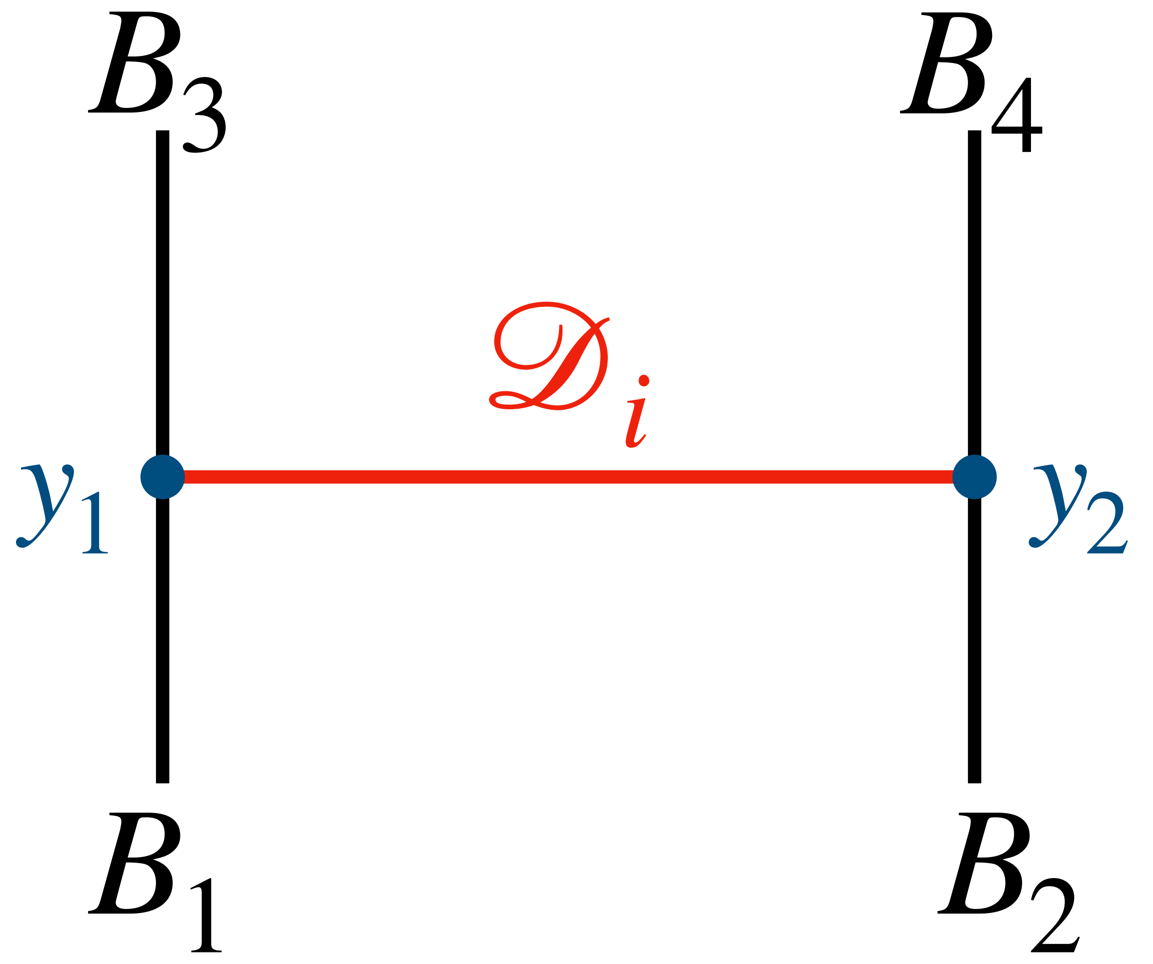
Global Symmetries in 2D CFTs

- Global (internal) symmetries in 2D CFTs are generated by topological line defects.
- They are also known as totally transmissive defects $[\mathscr{D}, L_n] = [\mathscr{D}, \bar{L}_n] = 0$.
- We focus on the case where we have a finite collection $\{\mathscr{D}_i\}$ of topological line defects closed under multiplication: $\mathscr{D}_i \times \mathscr{D}_j = \sum_k N_{ij}^k \mathscr{D}_k$.
- The collection of topological lines then forms a fusion category $\mathscr{C}$. [Ingo Runkel's talk]
- There is a plethora of examples, e.g., minimal models [Yu Nakayama's talk], WZW models, free boson [Yuma Furuta's talk], and so on.

Global Symmetries in 2D BCFTs



$$\mathcal{D}_i^{y_1 y_2} \equiv$$



$$: \mathcal{H}_{B_1 B_2} \to \mathcal{H}_{B_3 B_4}$$

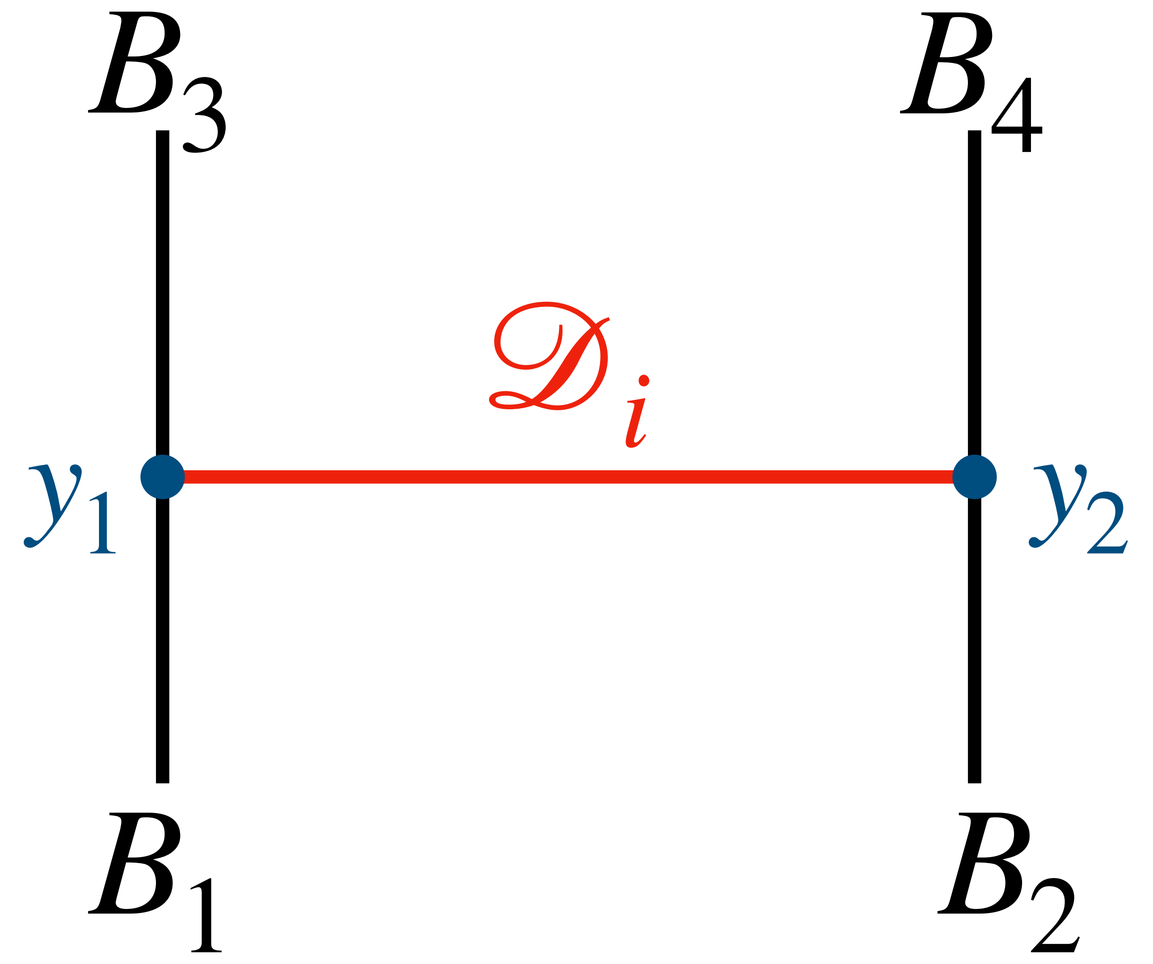
- We are interested in the symmetries of the entanglement Hamiltonian, or equivalently, of the open string Hamiltonian H_{open}.

- When a topological line $\mathcal{D}_i$ is stretched across the open interval, one has a freedom to choose the junction operators y_1 and y_2 at the endpoints.

Global Symmetries in 2D BCFTs

time ↑

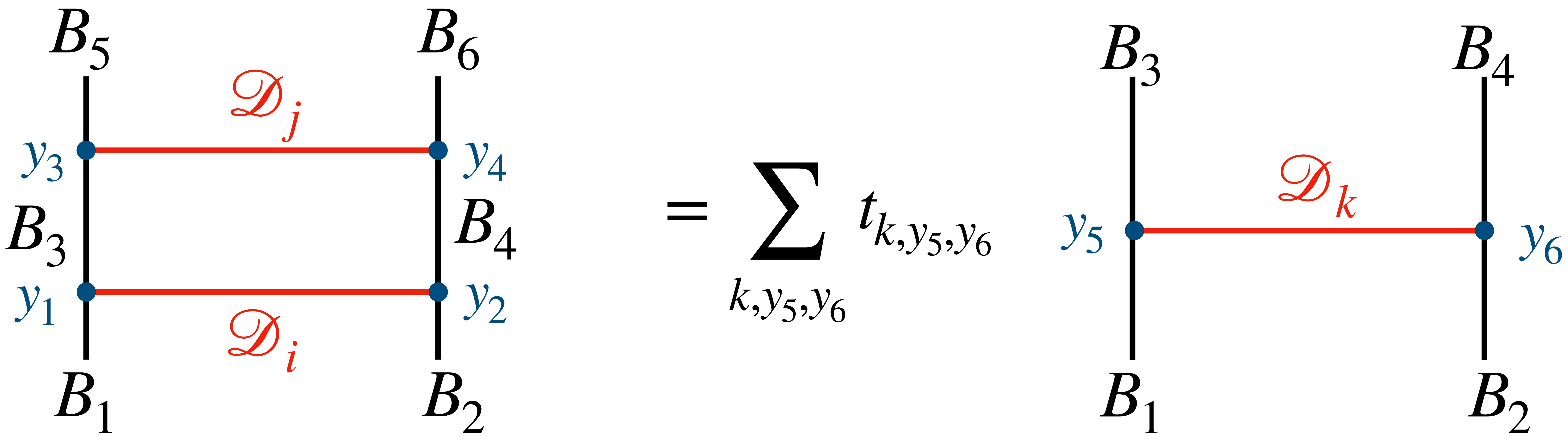
$\mathscr{D}_i^{y_1 y_2} \equiv$



$: \mathscr{H}_{B_1B_2} \to \mathscr{H}_{B_3B_4}$

- When the junction operators y_1 and y_2 are also chosen to be topological (zero scaling dimension), the whole configuration preserves the Virasoro symmetry.
- In particular, $\mathscr{D}_i^{y_1y_2} H_{open;B_1B_2} = H_{open;B_3B_4} \mathscr{D}_i^{y_1y_2}$ or $[\mathscr{D}_i^{y_1y_2}, H_{open}] = 0$ in short.

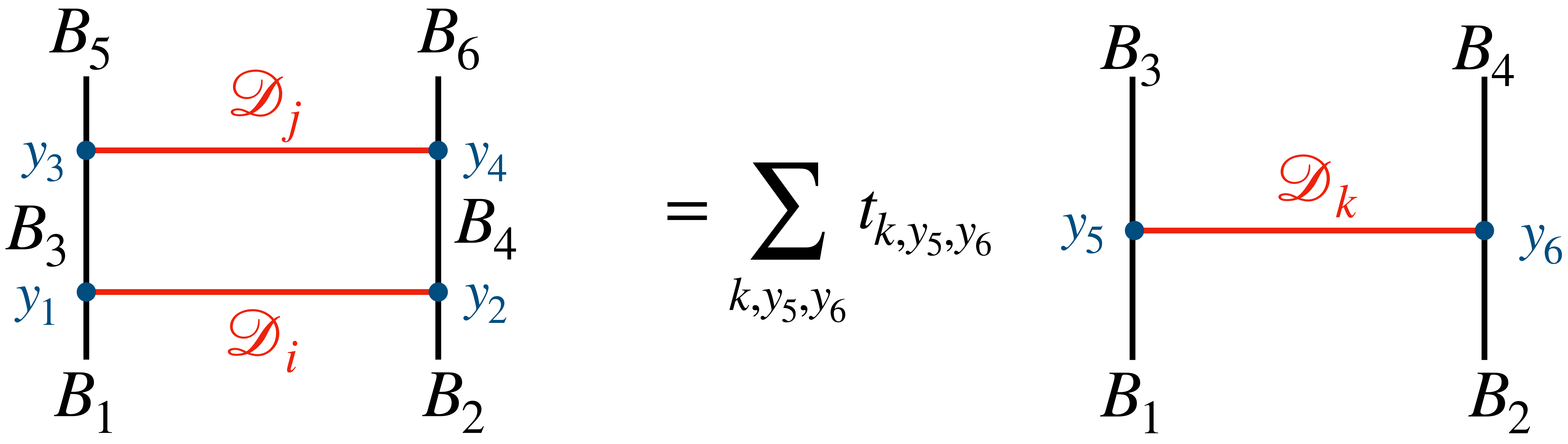
Global Symmetries in 2D BCFTs



The collection of operators $\{\mathscr{D}_i^{y_1 y_2}\}$ acts on the extended open string Hilbert space $\mathscr{H}_{open} \equiv \bigoplus_{BB'} \mathscr{H}_{BB'}$.

The operators can be multiplied: $\mathscr{D}_i^{y_1 y_2} \times \mathscr{D}_j^{y_3 y_4} = \sum_{k,y_5,y_6} t_{k,y_5,y_6} \mathscr{D}_k^{y_5 y_6}$ where $t_{k,y_5,y_6} \in \mathbb{C}$ can be computed in terms of the boundary 6j-symbols.

Global Symmetries in 2D BCFTs



- The collection of operators $\{\mathscr{D}_i^{ab}\}$ therefore defines a symmetry algebra acting on the open string Hilbert space.
- We call this algebra a boundary tube algebra (or strip algebra).
- "Boundary Tube Algebra = Symmetry of Entanglement Hamiltonian".
- Such an algebra also appears in many different physical contexts. [Kitaev-Kong; Inamura-Ohyama; Cordova-Holfester-Ohmori; Copetti-Cordova2-Komatsu; …]

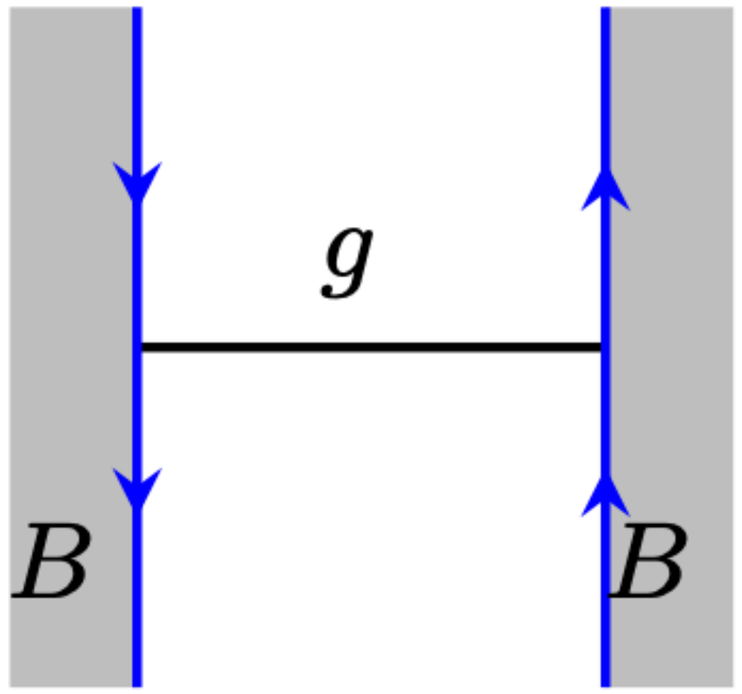
Boundary Tube Algebra

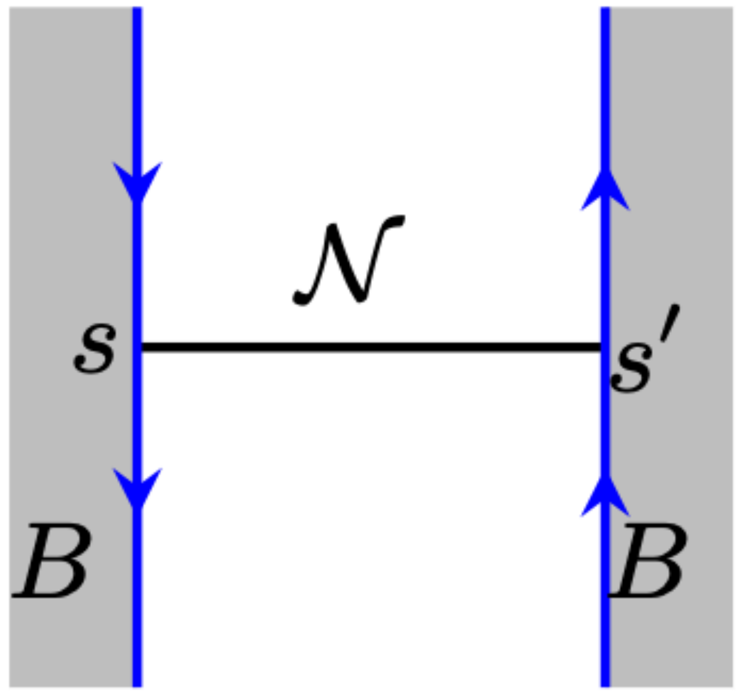
- The symmetry algebra in the open string channel so-defined is usually tedious to work out explicitly.
- However, it is often much easier to directly understand its representations.
- We will argue that the open string Hilbert space decomposes as

$$\mathscr{H}_{B_1B_2} = \bigoplus_{\alpha} \mathscr{H}^{\alpha}_{B_1B_2} = \bigoplus_{\alpha} \left(W^{\alpha}_{B_1B_2} \otimes \mathscr{V}^{B_1B_2}_{\alpha} \right) .$$

- α labels irreducible representations of the Boundary Tube Algebra.
- $W^{\alpha}_{B_1B_2}$ and $\mathscr{V}^{B_1B_2}_{\alpha}$ are certain 3D TQFT Hilbert spaces quantized on a disk with appropriately chosen boundary conditions and decorations by defects.

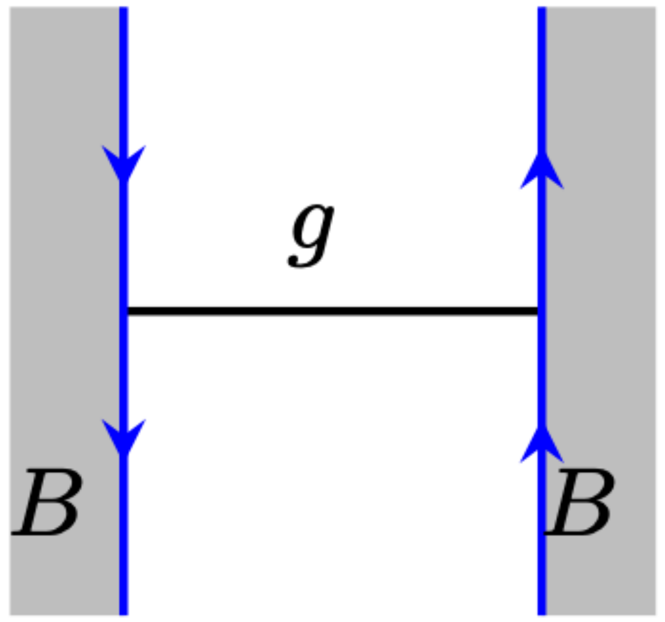
Example: Double Ising CFT

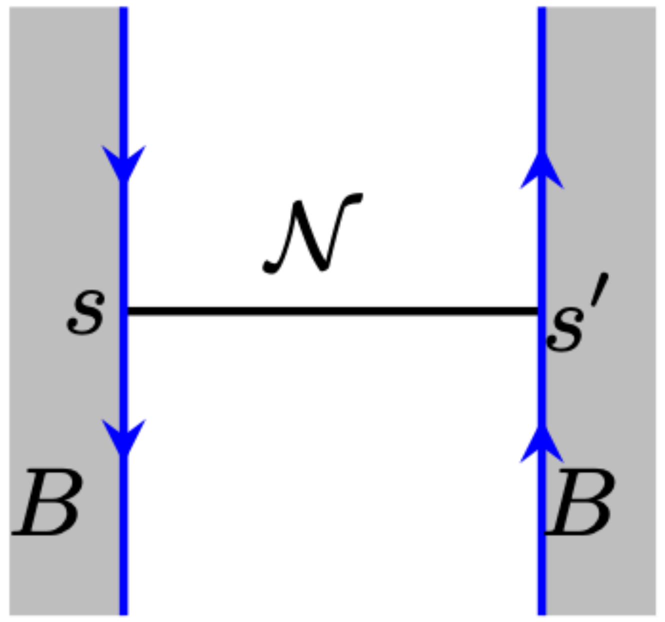
$\mathrm{H}_g =$ 

$\mathrm{H}_{\mathcal{N}}^{ss'} =$ 

- Consider two copies of the Ising CFT.
- The symmetry of interest will be the $\mathbb{Z}_2 \times \mathbb{Z}_2$ plus $\mathcal{N} \equiv \mathcal{N}_1 \mathcal{N}_2$.
- The double Ising CFT admits a (strongly) symmetric boundary condition B under this symmetry. [**YC**-Rayhaun-Sanghavi-Shao]

Example: Double Ising CFT

$\mathsf{H}_g =$ 

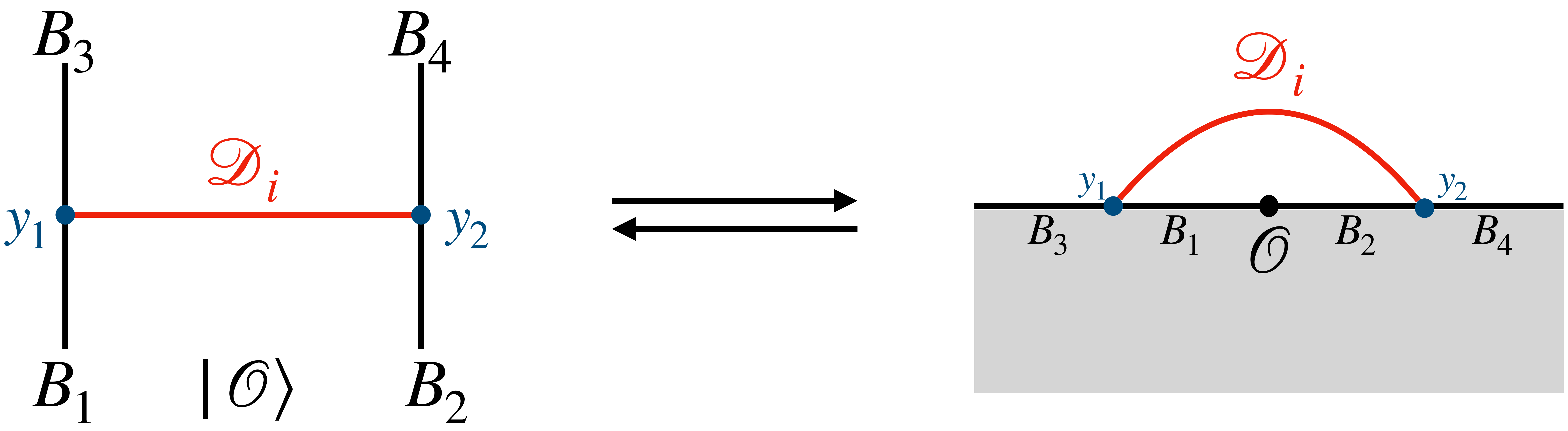
$\mathsf{H}_{\mathcal{N}}^{ss'} =$ 

$$
\begin{aligned}
\mathsf{H}_g \times \mathsf{H}_h &= \mathsf{H}_h \times \mathsf{H}_g = \mathsf{H}_{gh}\,, \\
\mathsf{H}_{\mathcal{N}}^{ss'} \times \mathsf{H}_a &= \mathsf{H}_b \times \mathsf{H}_{\mathcal{N}}^{ss'} = (-1)^s \mathsf{H}_{\mathcal{N}}^{s(s'+1)}\,, \\
\mathsf{H}_{\mathcal{N}}^{ss'} \times \mathsf{H}_b &= \mathsf{H}_a \times \mathsf{H}_{\mathcal{N}}^{ss'} = (-1)^{s'} \mathsf{H}_{\mathcal{N}}^{(s+1)s'}\,, \\
\mathsf{H}_{\mathcal{N}}^{s_1 s_2} \times \mathsf{H}_{\mathcal{N}}^{s_3 s_4} &= 2 \sum_{m,n=0,1} (-1)^{f_{mn}(s_i)} \mathsf{H}_{a^m b^n}\,,
\end{aligned}
$$

- The boundary tube algebra in this case is given by the Kac-Paljutkin Hopf algebra H_8.

- Historically, H_8 is the first Hopf algebra discovered which is neither commutative nor cocommutative.

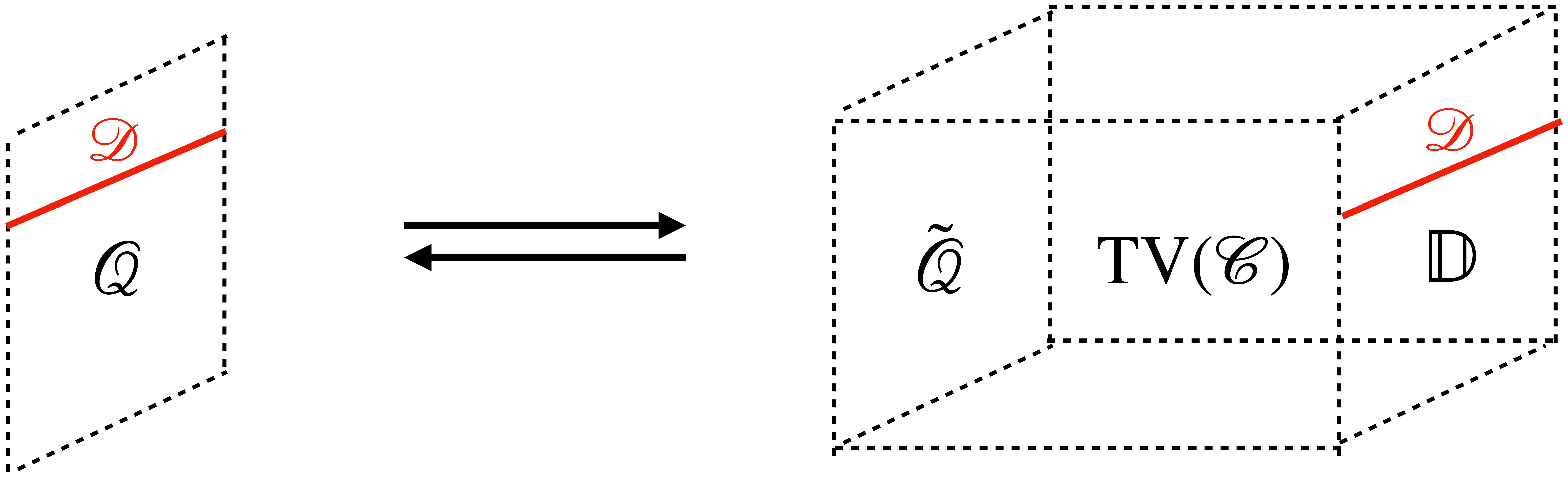
Boundary-Changing Operators



- Under the state-operator correspondence, the states in the open string Hilbert space $\mathscr{H}_{B_1 B_2}$ are mapped to boundary-changing local operators.
- Therefore equivalently, the spectrum of boundary-changing local operators organize into representations of the Boundary Tube Algebra.

Topological Holography / Symmetry TFT

Topological Holography

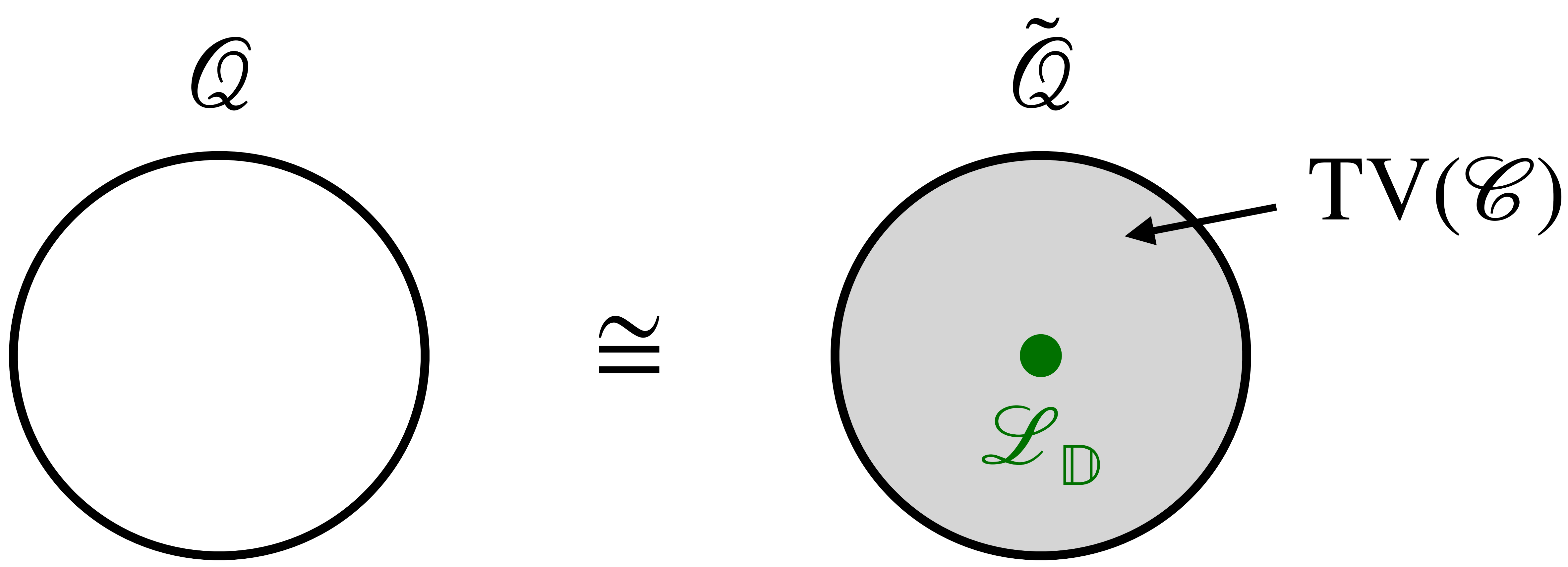


- The relation between 2D WZW models and 3D Chern-Simons TQFTs has a long history. [Moore-Seiberg; Witten; …; Fuchs-Schweigert-Runkel; …]

- More generally, any 2D QFT $\mathcal{Q}$ with a global symmetry can be uplifted to a boundary condition $\tilde{\mathcal{Q}}$ of a 3D Turaev-Viro TQFT. [Ingo Runkel's talk]

- The relation is sometimes referred to as "Topological Holography" or Symmetry TFT. […; Gaiotto-Kapustin-Seiberg-Willett; Gaiotto-Kulp; Freed-Moore-Teleman; Ji-Wen; …]

Topological Holography



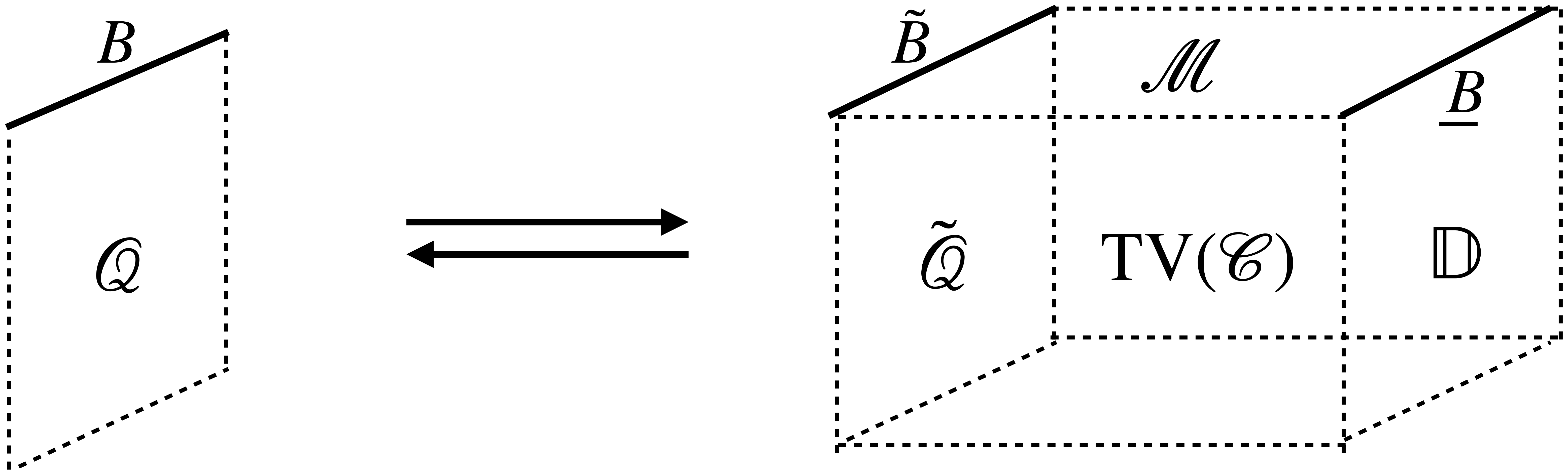
- It is generally interesting to fill in the "Holographic Dictionary."

- For instance, the closed string (circle) Hilbert space of a CFT $\mathcal{Q}$ is isomorphic to the disk D^2 Hilbert space of the Turaev-Viro TQFT punctured by a Lagrangian anyon $\mathscr{L}_{\mathbb{D}}$ and the boundary condition $\tilde{\mathcal{Q}}$ imposed on ∂D^2. [Elitzur-Moore-Schwimmer-Seiberg; …]

Topological Holography of Conformal Boundaries

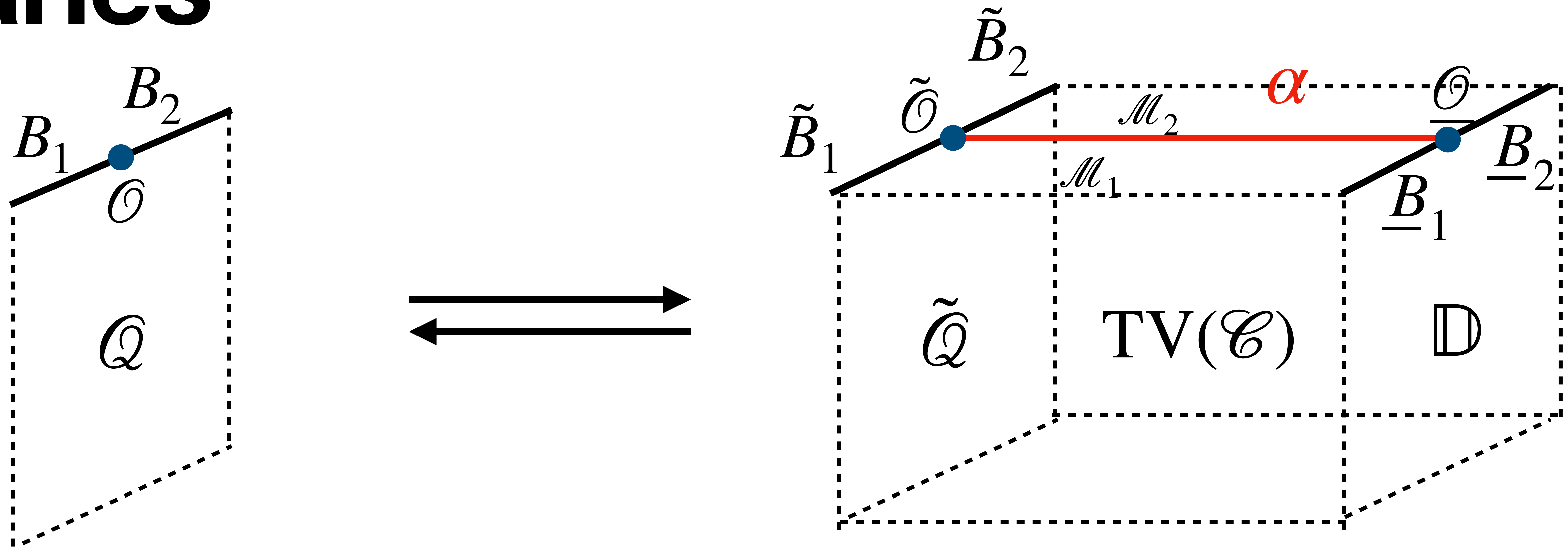
- What is the 3D "bulk dual" of conformal boundary conditions of a 2D CFT $\mathscr{Q}$?
- Suppose $\mathscr{Q}$ has fusion category symmetry $\mathscr{C}$ and a multiplet of conformal boundaries $\{B_i\}$ transforming into each other under the action of $\mathscr{C}$.
- Mathematically, the set of conformal boundary conditions $\{B_i\}$ forms a module category $\mathscr{M}$ over the fusion category $\mathscr{C}$.
- Moreover, it is known that topological boundary conditions of $\mathrm{TV}(\mathscr{C})$ are in 1-to-1 correspondence with module categories $\mathscr{M}$. [Kitaev-Kong]

Topological Holography of Conformal Boundaries



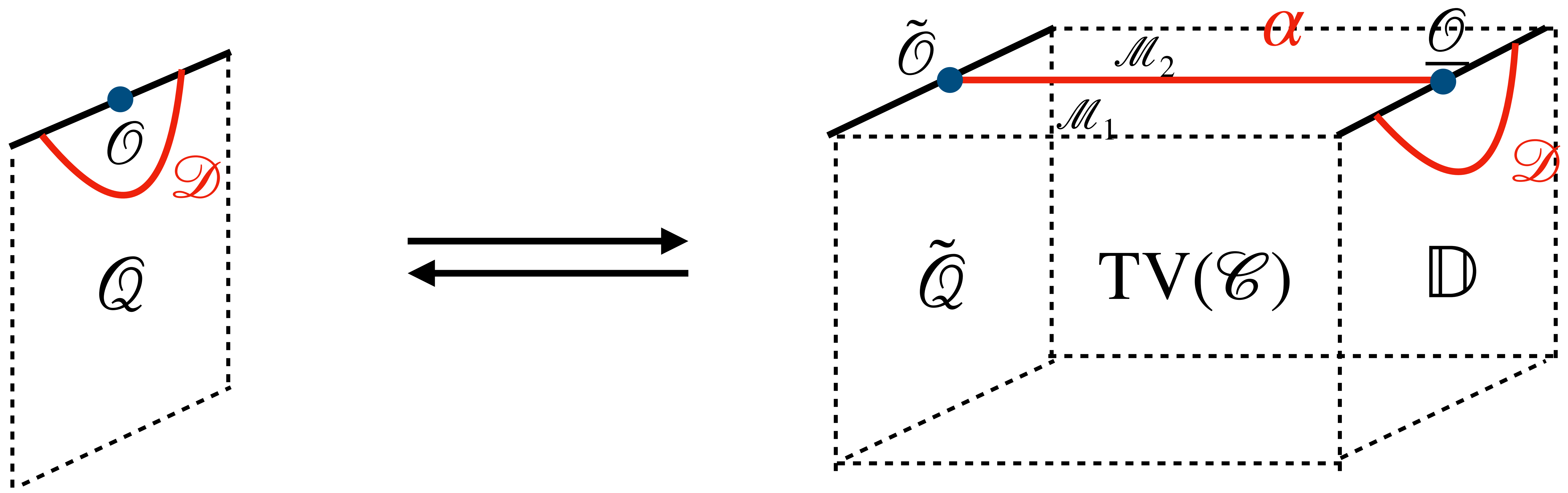
- We claim that a conformal boundary condition B in 2D inflates into a topological boundary condition of the 3D TQFT, labeled by the module category $\mathscr{M}$ that it belongs to. [**YC**-Rayhaun-Zheng; Huang-Cheng; Cordova-Holfester-Ohmori; Copetti-Cordova2-Komatsu; Bhardwah-Copetti-Pajer-SchaferNameki].

Topological Holography of Conformal Boundaries



- Let $\mathcal{O}$ be a boundary-changing local operator between two conformal boundaries B_1 and B_2, belonging to module categories $\mathcal{M}_1$ and $\mathcal{M}_2$.
- $\mathcal{O}$ becomes a triple $(\underline{\mathcal{O}}, \alpha, \tilde{\mathcal{O}})$ on the 3D side.
- α is a line interface between two TQFT boundary conditions labeled by $\mathcal{M}_1$ and $\mathcal{M}_2$.

Topological Holography of Conformal Boundaries



α labels representations of the boundary tube algebra. Action of a topological line $\mathscr{D}$ never changes α on the 3D side. (Mathematically α is given by module functors from $\mathscr{M}_1$ to $\mathscr{M}_2$.) [Douglas-SchommerPries-Snyder; Barter-Bridgeman-Wolf; Bai-Zhang (this week)]

The open string Hilbert space decomposes as $\mathscr{H}_{B_1B_2} = \bigoplus_\alpha \mathscr{H}^\alpha_{B_1B_2}$.

Non-Invertible Symmetry and Entanglement Entropy

Symmetry-Resolved Entanglement Entropy

- We have understood the global symmetry of the Entanglement Hamiltonian $H_{EE} = -2\pi \log \rho_A$ (open string Hamiltonian): Boundary Tube Algebra.
- We have understood the representations: Boundary Line Interfaces α in 3D TQFT. (Mathematically, module functors.)
- As an immediate application, we can compute the Non-Invertible Symmetry-Resolved Entanglement Entropy.

Symmetry-Resolved Entanglement Entropy

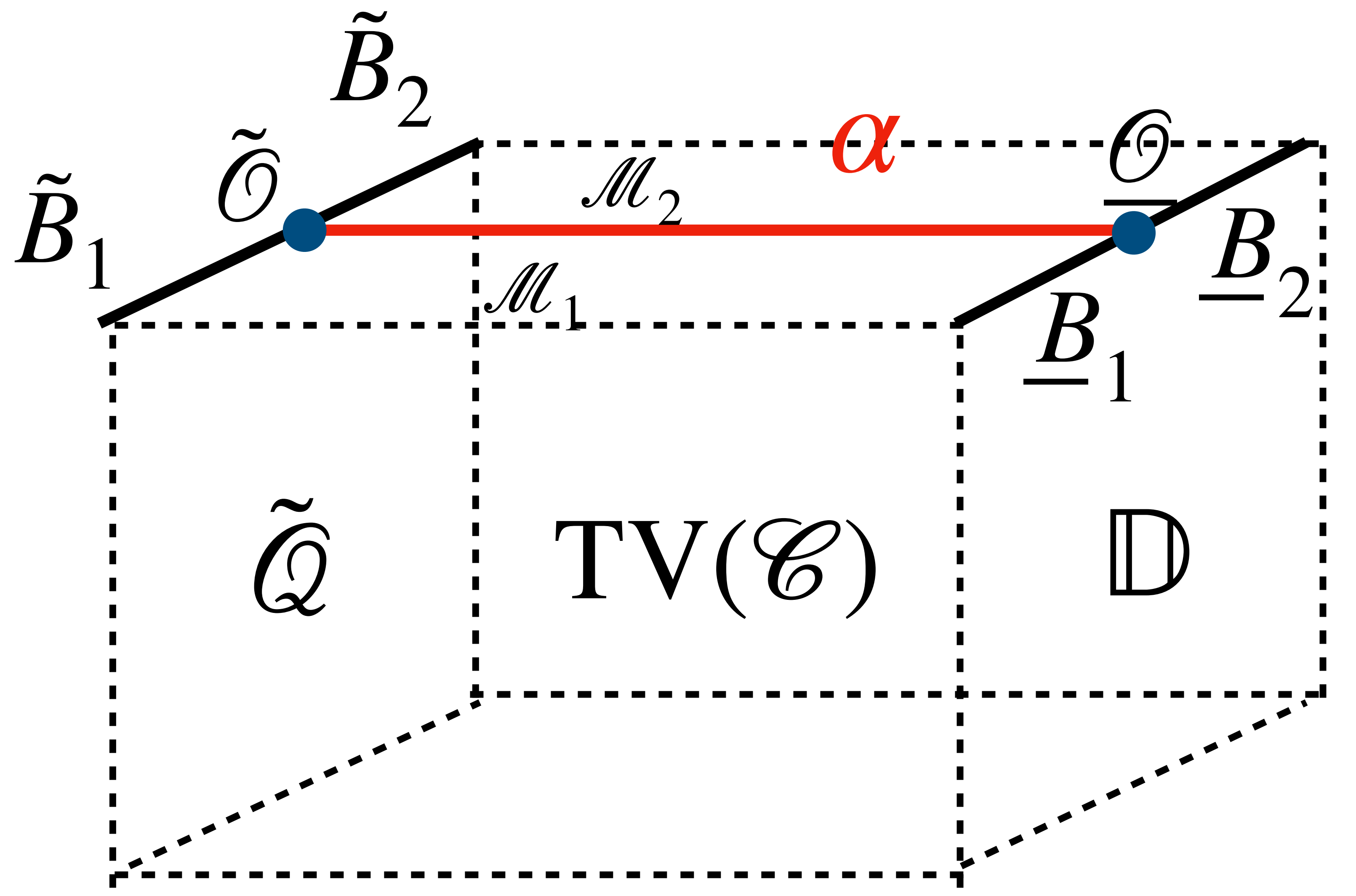
- Using the symmetry of the Entanglement Hamiltonian, we block diagonalize the reduced density matrix ρ_A into different representation sectors:

$$\rho_A = \bigoplus_{\alpha} \rho_A^{\alpha}.$$

- The Symmetry-Resolved Entanglement Entropy is defined to be the entropy of each block:

$$S_{EE}^{\alpha} = -\,\mathrm{Tr}\tilde{\rho}_A^{\alpha}\log\tilde{\rho}_A^{\alpha}.$$

Symmetry-Resolved Entanglement Entropy



Using the correspondence to 3D TQFT, we can compute

$$S_{EE}^{\alpha} = \frac{c}{3}\log\frac{L}{\epsilon} + \log(g_1 g_2) + \log\frac{d_\alpha N_{\alpha B_1}^{B_2}}{d_{B_1} d_{B_2}}.$$

$$\lim_{\epsilon\to 0}[S_{EE}^{\alpha} - S_{EE}] = \log\frac{d_\alpha N_{\alpha B_1}^{B_2}}{d_{B_1} d_{B_2}}.$$

Symmetry-Resolved Entanglement Entropy

$$\lim_{\epsilon\to 0}[S^{\alpha}_{EE} - S_{EE}] = \log\frac{d_\alpha N^{B_2}_{\alpha B_1}}{d_{B_1} d_{B_2}}$$

- For ordinary (finite) symmetry G, and symmetric boundary conditions, we obtain $\log\frac{d_\alpha^2}{|G|}$, reproducing the results of [Casini-Huerta-Magan-Pontello; Magan; Kusuki-Murciano-Ooguri-Pal; see also Harlow-Ooguri; Pal-Sun].
- For non-invertible symmetries, and (strongly) symmetric boundaries, we obtain $\log\frac{d_\alpha^2}{\dim(\mathscr{C})^2}$. Similar results were reported in [Benedetti-Casini-Kawahigashi-Longo-Margan; SauraBastida-Das-Sierra-MolinaVilaplana; see also Lu-Sun; Lin-Okada-Seifnashri-Tachikawa].
- The result for completely symmetry breaking boundaries were derived in [Heymann-Quella] for rational CFTs, together with some numerical tests.

Symmetry-Resolved Entanglement Entropy

- For (strongly) symmetric boundaries at the entanglement cut

$$\lim_{\epsilon\to 0}[S^{\alpha}_{EE} - S_{EE}] = \log\frac{d_\alpha^2}{\dim(\mathscr{C})^2}.$$

- Every irreducible representation α appears in the spectrum. "Completeness of the entanglement spectrum."

- For the trivial representation $\alpha = 1$, we find $\log\frac{1}{\dim(\mathscr{C})^2}$ = Topological Entanglement Entropy of the 3D Turaev-Viro TQFT. [Kitaev-Preskill; Levin-Wen]

Summary

- In 2D CFTs, the entanglement of a single interval can be analyzed using conformal boundaries.
- The Entanglement Hamiltonian has Boundary Tube Algebra as a global symmetry.
- We computed the corresponding Symmetry-Resolved Entanglement Entropy. The formula reproduces several known results, and generalizes them.
- Underlying this, the correspondence to 3D TQFT was crucial.

Thank you!