Field Theory with $\mathbb{Z}_N$ 1-form Symmetry dual to $SU(N)$ Gauge Theory

Kiyoharu Kawana (KIAS)

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See also arXiv: 2310.07993 (collaboration with Prof. Yoshimasa Hidaka (YITP))

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What is higher-form global symmetry ?

- Ordinary symmetry = 0-form symmetry i.e. Symmetry operator acts on a point

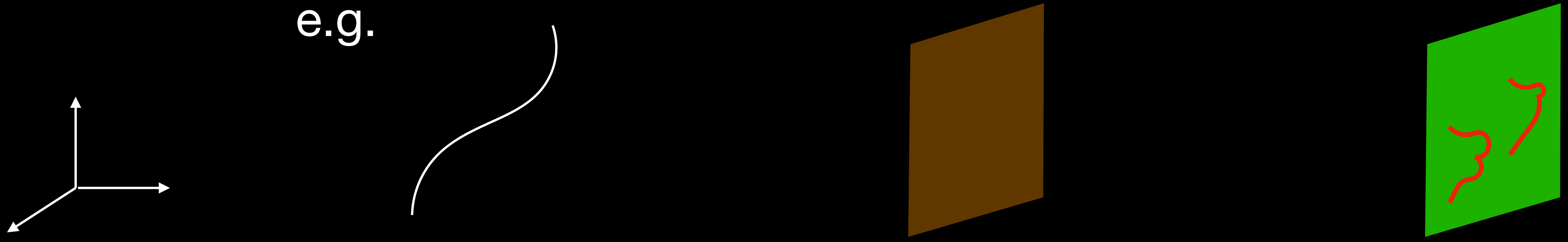
  From Noether theorem, we have a conservation law i.e. conserved current J^μ

$$\partial_\mu J^\mu = 0 \qquad \text{or} \qquad d \star J_1 = 0 \quad (J_1 = J_\mu dx^\mu) \qquad \text{differential form}$$

- But, we can also consider higher dimensional objects extended in space(time)

e.g.

Space



Vortex　　Domain wall　　p brane

Higher-form symmetry = Global symmetry of extended objects

[Gaiotto, Kapstin, Seiberg, Willet ('15), …]

- Natural generalization of conservation law to p-dimensional object is

$$p = 0 \qquad\qquad p > 0$$

$$d \star J_1 = 0 \quad \longrightarrow \quad d \star J_{p+1} = 0$$

- Symmetry (topological) operator is given by

$$Q_p = \int_{\Sigma_{D-p-1}} \star J_{p+1} \, , \qquad U_\theta(\Sigma_{D-p-1}) = \exp\left(i\theta Q_p\right)$$

Conserved charge

Symmetry operator

Σ_{D-p-1} = $(D-p-1)$-dimensional Subspace

$D =$ spacetime dimension

Higher-form symmetries in QFT

- Ex 1. Abelian-Higgs model (with a charge q) → 1-form electric $\mathbb{Z}_q$ symmetry

$$U_1(S) = \exp\left(\frac{i}{q}\int_S \star F_2\right) \quad \text{(symmetry operator)} \qquad W(C) = \exp\left(i\oint_C A\right) \quad \text{(charged operator)}$$

In the Higgs phase, another $\mathbb{Z}_q$ 2-form symmetry emerges $\quad B_2 \rightarrow B_2 + \frac{1}{q}\Lambda_2, \quad d\Lambda_2 = 0$

Low-energy effective action $\quad S_{\text{eff}} = \frac{q}{2\pi}\int_{\Sigma_4} B_2 \wedge dA_1 \quad$ Exhibits topological order

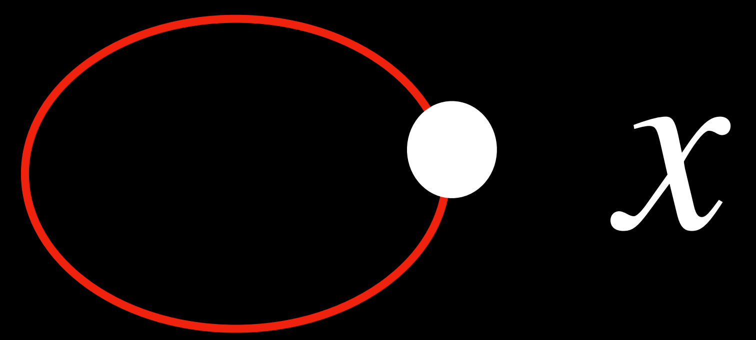
Ex 2. $\mathrm{SU}(N)$ pure (lattice) Yang-Mills theory

$$S_{\mathrm{YM}} = -\frac{1}{2g^2}\int_{\Sigma_D} \mathrm{Tr}\,(G \wedge \star G) \ , \quad G = dA - iA \wedge A$$

The gauge fields are blind to the center group $\mathbb{Z}_N$ of $\mathrm{SU}(N)$

→ We can consider a twisted (gauge) transformation $g(x) \in \mathrm{SU}(N)$ such as

$$g(x) \to e^{\frac{2\pi i}{N}} g(x) \quad \text{along a loop}$$

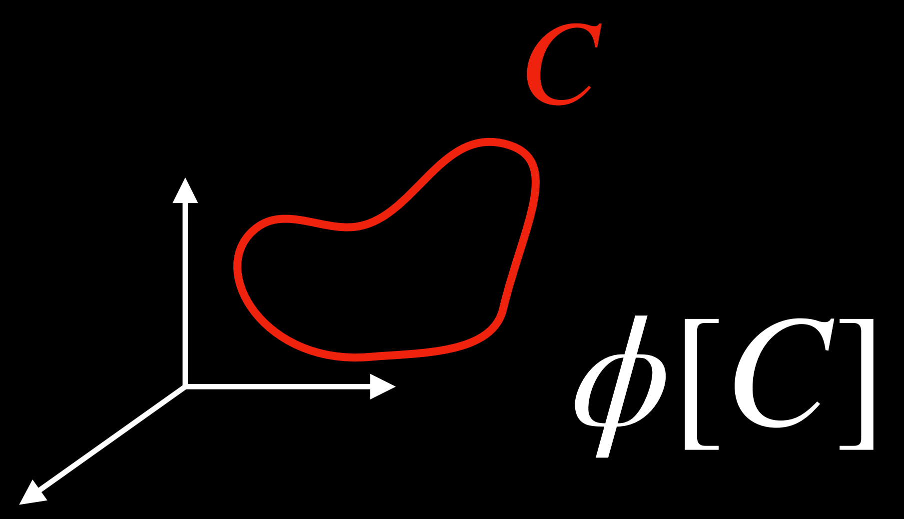


→ Action is invariant under this transformation = $\mathbb{Z}_N$ 1-form global symmetry

The charged object is Wilson Loop

$$W[C] = \mathrm{Tr}\left[P \exp\left(i\oint_C A\right)\right] , \qquad W[C] \ \to \ e^{\frac{2\pi i}{N}} W[C]$$

Summary of my talk



1. I will derive a scalar field theory dual to $\mathrm{SU}(N)$ (lattice) gauge theory

   $\Longrightarrow$ The resultant theory is a field theory on loop space C = a String Field Theory (SFT)

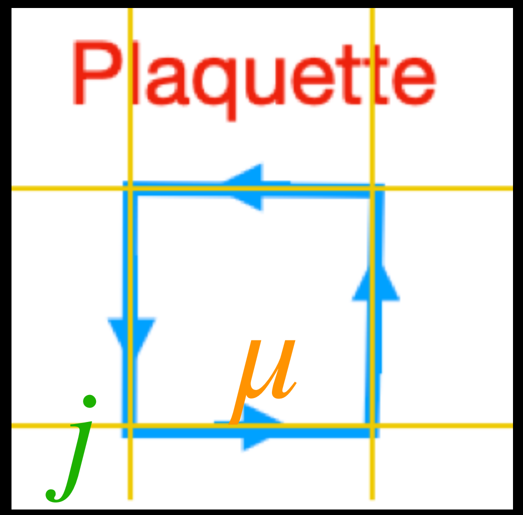
2. and clarify the structure of the $\mathbb{Z}_N$ 1-form global symmetry in the dual SFT

   Dual SFT = Landau theory with $\mathbb{Z}_N$ 1-form global symmetry

3. perform the mean-field analysis of the dual SFT

And found {
- Area-law behavior of the string field
- Non-trivial topological defect in the broken phase
- Topological order in the broken phase (i.e. BF-type topological field theory)

Lattice Gauge Theory (Quick Review)



- Partition function $\qquad Z = \int [dU] e^{-S_E[U]}, \qquad S_E = \beta \sum_P \left[1 - \frac{1}{2N} (W[P] + W^\dagger[P]) \right], \qquad \beta = \frac{2N}{g^2}$

$A_\mu(j) \in$ Lie algebra of $\mathrm{SU}(N)$, $\quad W[P] = \mathrm{Tr}\left[\prod_{(\mu,j)\in P} U_\mu(j) \right] = \mathrm{Tr}\left[e^{ia \sum_{(\mu,j)\in P} A_\mu(j)} \right]$ $\quad$ (Wilson loop)

1. Defined on a (D-dimensional) cubic lattice Λ_D with a lattice size a
2. $\mathrm{SU}(N)$ gauge symmetry $\quad U_{\hat{\mu}}(j) \quad \to \quad g(j) U_{\hat{\mu}}(j) g(j+\hat{\mu}) \ , \quad g(j) \in \mathrm{SU}(N)$
3. It has $\mathbb{Z}_N$ 1-form global symmetry (center symmetry)

$$W[C] \quad \to \quad e^{2\pi i \frac{m}{N}} W[C]$$

Dual transformation

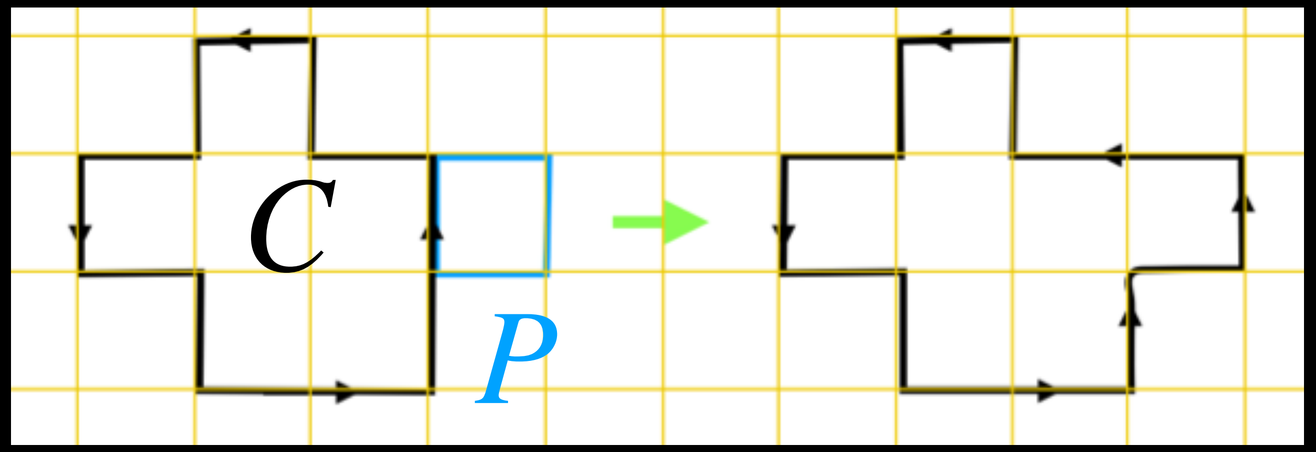
[Yoneya, Banks, Kawai, Polyakov … in 80's]

- We can rewrite the lattice action as a sum over all loops C on lattice by introducing the operator of adding a plaquette:

$$\hat{\Pi}_P U[C] := U[C+P]$$

Path-ordered product

$$U[C] = \prod_{(\mu,i)\in C} U_\mu(i)$$



where $C+P$ means a combined loop by erasing the common links

- After some calculations, we can show that the action becomes quadratic form

$$S_E = \sum_C \mathrm{Tr}[U^\dagger[C]\hat{H}U[C]]$$

⇐ This can be rewritten as a gaussian path-integral

(= Hubbard-Stratonovich transformation)

where

$$\hat{H}U[C] = \frac{1}{2(D-1)|C|}\sum_{P\in C}\hat{\Pi}_P U[C]$$

Dual SFT and $\mathbb{Z}_N$ 1-form symmetry

- The dual SFT is

$$Z = \int [d\phi] \exp\left(-\sum_{C} \mathrm{Tr}\left(\phi^{\dagger}[C] \hat{H}^{-1} \phi[C] \right) - V[\phi^{\dagger}\phi] \right)$$

$\phi[C] =$ complex $N \times N$ matrix string field, $\hat{H}^{-1}$ is the inverse operator of $\hat{H}$

- The potential is determined by

$$V[\phi^{\dagger}\phi] = -\log\left[\int [dU] \exp\left(-\sum_{C} \mathrm{Tr}(U^{\dagger}\phi[C]) + \mathrm{h.c.} \right) \right]$$

The path-integral of original link variables

- In this formulation, $\mathbb{Z}_N$ 1-form global symmetry is quite manifest

$$\phi[C] \quad \to \quad e^{2\pi i \frac{m}{N}} \phi[C]$$

just like the ordinary QFT

Classical continuum limit

- We can also take the classical continuum limit in the dual SFT

$$Z = \int [d\phi] \exp\left(-\sum_{C} \mathrm{Tr}\left(\phi^\dagger[C]\hat{H}^{-1}\phi[C]\right) - V[\phi^\dagger\phi] \right)$$

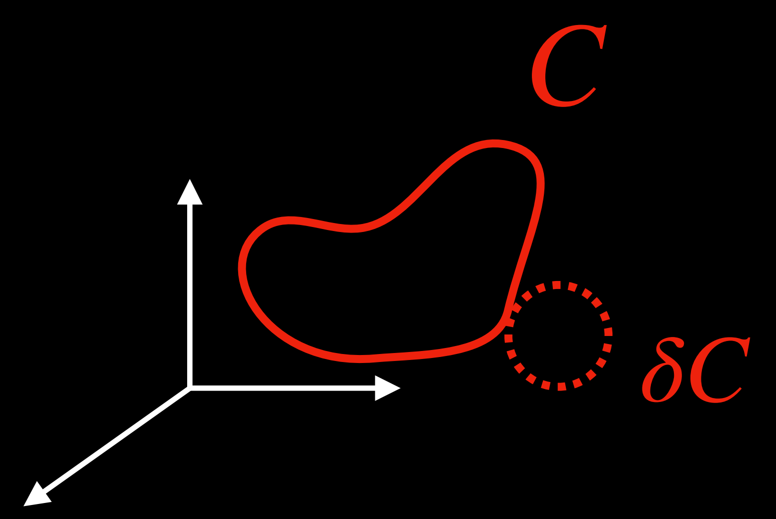
$a \to 0$

$$\int [d\phi] \exp\left(-\int \mathscr{D}X \left[\frac{1}{L[C]} \int_0^{2\pi} d\xi \sqrt{h(\xi)}\, \mathrm{Tr}\left(\phi^\dagger[C] \frac{\delta^2}{\delta S^\mu(\xi) S_\mu(\xi)} \phi[C] \right) \right] - V[\phi^\dagger\phi] \right)$$

Here

- ξ is the intrinsic parameter of C, and $L[C]$ is the length of C
- $\dfrac{\delta}{\delta S^\mu(\xi)}$ is a functional derivative, known as Area derivative

[Migdal and Makeenko (80), Polyakov (80), KK and Hidaka (23)]



Small deformation of loop

Let’s perform the mean-field analysis in this dual SFT.

In particular, how is the SSB of 1-form symmetry described ?

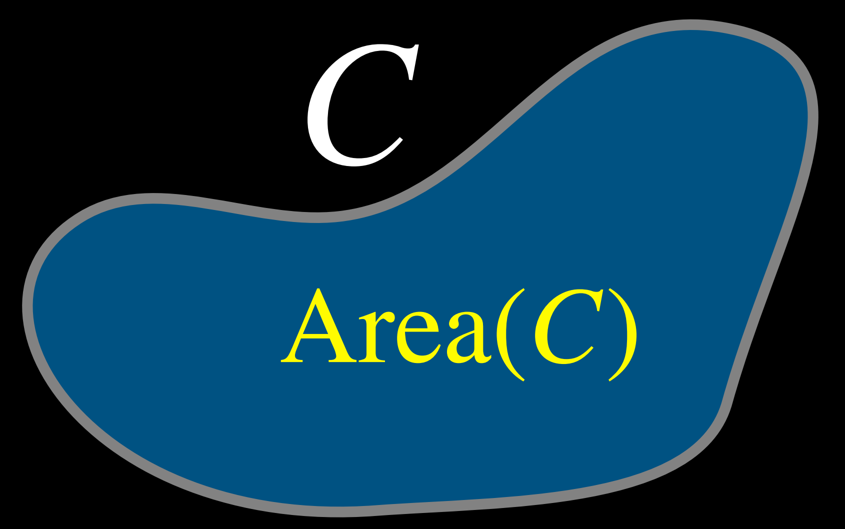
$$\langle \phi[C_p] \rangle = ?$$

The potential is

$$V(\phi) = \mu \mathrm{Tr}(\phi^*[C]\phi[C]) + \frac{\lambda}{4}\mathrm{Tr}(\phi^*[C]\phi[C])^2 + \cdots$$

Unbroken phase ($\mu > 0$) $\leftrightarrow$ Area law

- Consider the following ansatz: $\phi[C] = \frac{1}{\sqrt{2}} f(\text{Area}(C)) \times I$



$\text{Area}(C)$ is the minimal surface-area surrounded by C

- Then, the EOM of $f(z)$ is given by

$$f''(z) - q(z) f'(z) - \frac{\partial V}{\partial f} = 0 \ , \quad z = \text{Area}(C) \ , \qquad \text{with BC: } f(z = \infty) = 0$$

- $q(z)$ should behave as $q(z) \propto 1/z$ for $z \to \infty$ by dimensional analysis

$$\therefore \text{ For large } z \qquad \underset{\mu > 0}{f''(z) - \mu f(z) \simeq 0} \quad \Longrightarrow \quad f(z) \underset{z \to \infty}{\simeq} e^{-\sqrt{\mu} \times \text{Area}(C)} \qquad \text{(Area law)}$$

Broken phase ($\mu < 0$) ↔ Perimeter law

- The broken phase is simply described by the VEV $\langle \phi[C] \rangle = v \neq 0$ as in ordinary QFT

- In the context of higher-form symmetry, this is often referred to the Perimeter law

$$\langle \phi[C] \rangle \sim e^{-cL[C]} \,, \quad L[C] = \text{length of C}$$

This exponential behavior can be "renormalized" by the redefinition of $\phi[C]$

- Now one can see that string field $\phi[C]$ corresponds to the Wilson loop of original gauge theory

$$\phi[C] \sim W[C] \sim \begin{cases} e^{-\sqrt{\mu} \times \text{Area}(C)} & \text{(Unbroken phase)} \\ v \neq 0 & \text{(Broken phase)} \end{cases}$$

Low-energy effective mode

- As in ordinary QFT, we can discuss low-energy fluctuation modes
- A natural candidate is the phase modulation

$$\phi[C_1] = \frac{v}{\sqrt{2}} \exp\left(i \int_{C_1} A_1 \right) , \quad A_1 = 1 - \text{form field}$$

1-form transformation is

$$A_1 \to A_1 + d\Lambda$$

- If 1-form symmetry were $\mathrm{U}(1)$, A_1 is gapless and the effective action is

$$S[A_1] = -\frac{v^2}{2} \int_{\Sigma_D} F_2 \wedge \star F_2 \, , \quad F_2 = dA_1$$

This is nothing but the Maxwell theory ⟹ Photon appears as NG mode

Gapped mode and topological field theory

- In the present dual SFT, however, 1-form symmetry is $\mathbb{Z}_N$ $\Longrightarrow$ Effective theory must be gapped
- In fact, we can calculate the effective action and find BF-type topological field theoy

$$S_{\rm eff} \sim -\frac{N}{2\pi}\int_{\Sigma_D} B_{D-2} \wedge dA_1$$

$B_{D-2} = (D-2)$-form auxiliary field

- In addition to the original 1-form symmetry $A_1 \to A_1 + \frac{1}{N} d\Lambda,$ there is an emergent $(D-2)$-form symmetry

$$B_{D-2} \quad \to \quad B_{D-2} + \frac{1}{N} d\Lambda_{D-3}$$

This is related to the existence of Topological defect

Topological defect

Axion potential $V(\phi + 2\pi) = V(\phi)$

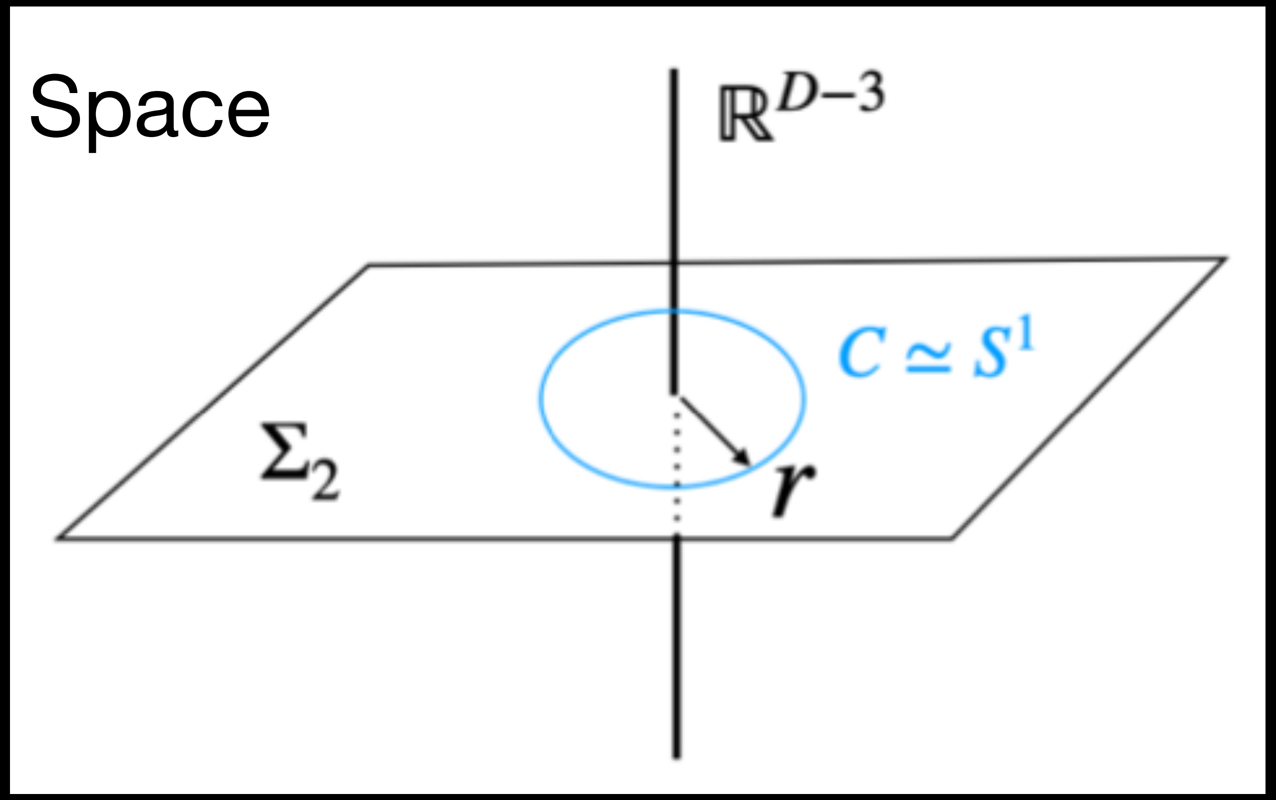
- For example, for an ordinary $\mathbb{Z}_N$ 0-form symmetry, the low-energy mode is axion $\phi(x)$

$\Longrightarrow$ Topological defect is Domain-wall

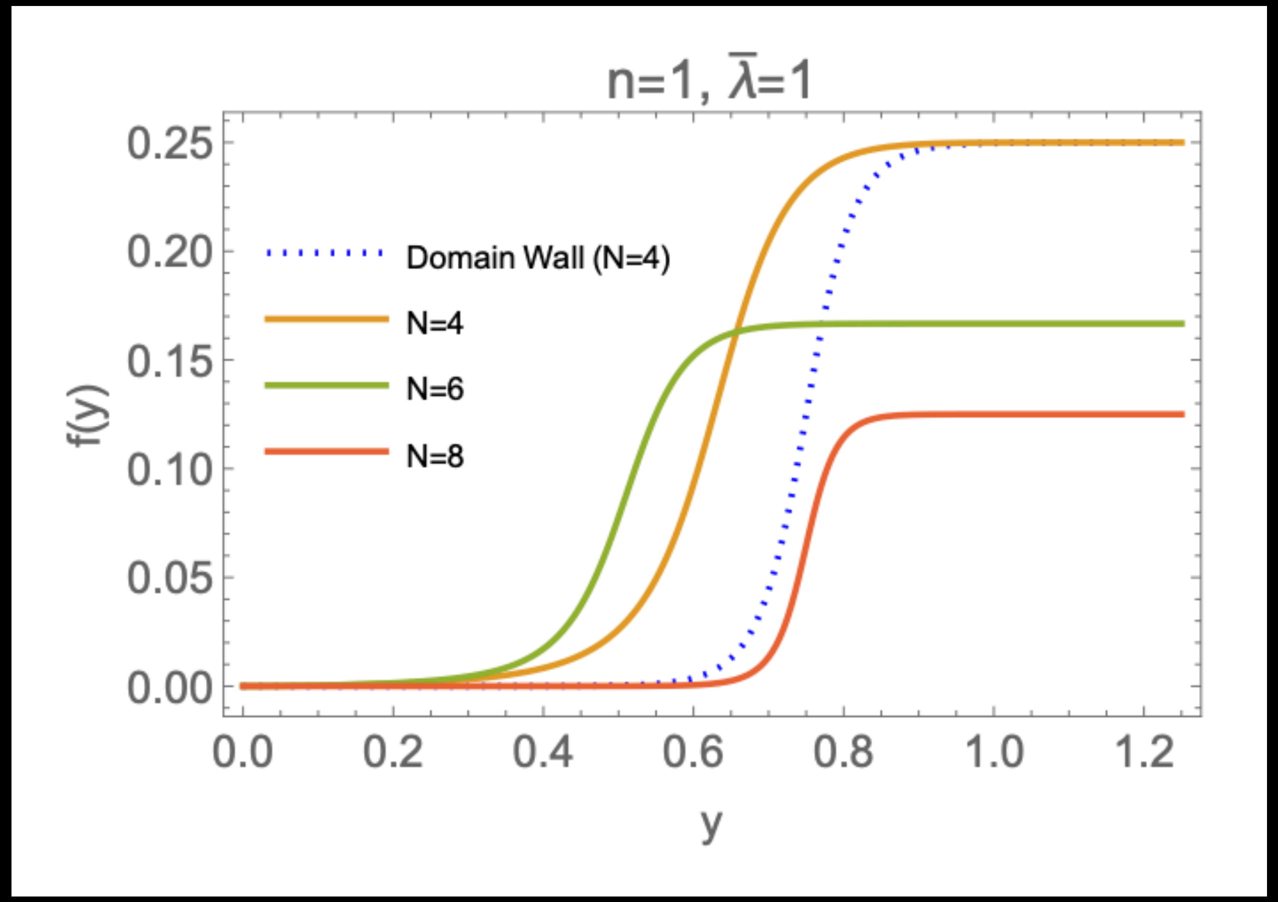
- In the present $\mathbb{Z}_N$ 1-form symmetry, we can similarly construct a $(D-3)$-dimensional static configuration (aka Center-Vortex configuration)

ansatz

$$\phi[C] = \frac{v}{\sqrt{2}} \exp\left(2\pi i f(r)\right)$$



C is embedded as S^1 with radius r



Numerical plots of $f(r)$

Degerate vacua are interpolated like domain wall

Topological order

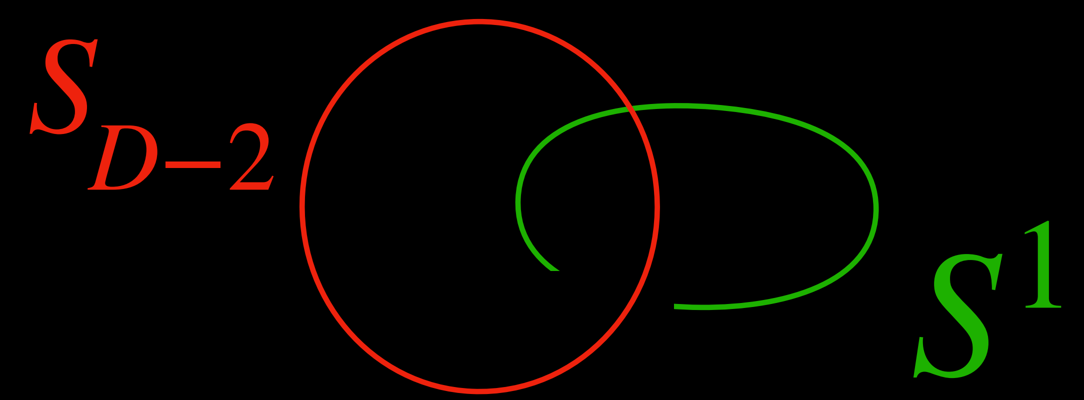
$$S_{\text{eff}} \sim -\frac{N}{2\pi}\int_{\Sigma_D} B_{D-2}\wedge dA_1$$

$$W[C] = e^{i\int_C A_1}\ ,\quad V[C_{D-2}] = e^{i\int_{C_{D-2}} B_{D-2}}$$

- For $\Sigma_D = \mathbb{R}\times\mathbb{R}^{D-1}$, they trivially act on the ground state since

$$\langle W[C]\rangle = 1\ ,\quad \langle V[C_{D-2}]\rangle = 1$$

Space with non-trivial topology



- On the other hand, they can act non-trivially when Σ_{D-1} is topologically nontrivial

e.g. When $\Sigma_D = \mathbb{R}\times S^1\times S^{D-2}$, by taking $C = S^1$, $C_{D-2} = S^{D-2}$

$$V[C_{D-2}]W[C]V[\overline{C}_{D-2}] = e^{\frac{2\pi i}{N}}W[C]$$

as an equal-time operator relation

By using this relation, we can show that there are N degenerate ground states

$$|\Omega\rangle\ ,\quad V[C_{D-2}]\,|\Omega\rangle\ ,\quad V[C_{D-2}]^2\,|\Omega\rangle\ ,\ \cdots\ ,\quad V[C_{D-2}]^{N-1}\,|\Omega\rangle\ ,$$

Summary of my talk

1. I derived a dual scalar field theory of $\mathrm{SU}(N)$ (lattice) gauge theory

$$\int [d\phi]\exp\left(-\int \mathscr{D}X\left[\frac{1}{L[C]}\int_0^{2\pi} d\xi\sqrt{h(\xi)}\mathrm{Tr}\left(\phi^\dagger[C]\frac{\delta^2}{\delta S^\mu(\xi)S_\mu(\xi)}\phi[C]\right)\right]-V[\phi^\dagger\phi]\right)$$

This theory has a $\mathbb{Z}_N$ 1-form global symmetry manifestly

2. I performed the mean-field analysis of the dual SFT and found
   - Area-law behavior of the string field
   - Non-trivial topological defect in the broken phase
   - Topological order in the broken phase (i.e. BF-type topological field theory)

3. Can we say more about non-perturbative aspects of gauge theory ? (Future works)