Foundations of Asymptotics and Perturbation Theory

An Outside View

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Disclosure

- I am a mathematical physicist with **no** expertise in astrophysics, perturbation theory or the use of asymptotic methods...
- I have no expertise on most of the topics I will cover in my talk today.

Part I

Prolegomena

What is Perturbation Theory?

A Preliminary Characterisation:

Perturbation theory deals with systematic methods of organising small corrections/deformations to well understood (solvable?) models/objects.

Examples:

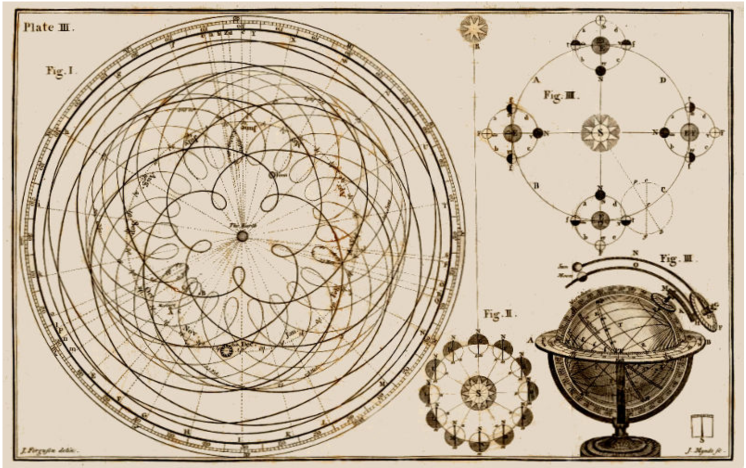
- Physics:
  - Celestial mechanics.
  - Theory of waves.
  - QM (Rayleigh-Schrödinger).
  - QFT (+ renormalisation).
  - Adiabatic continuity.
  - Renormalisation group.
  - ...
- Mathematics:
  - Formal solutions to differential equations.
  - Analysis of linear operators.
  - Stability, Catastrophe and Deformation theory.
  - Dynamical systems/KAM theory.
  - ...

Synopsis by way of a Quote:

Barry Mazur, *Perturbations, Deformations and Variations (and "Near Misses") in Geometry, Physics and Number Theory*

Whatever it leads to, the perturbative strategy is everywhere in mathematics and takes many forms. ... Some questions become meaningful only when they are treated as specific instances within a field of closely related questions. Often the landscape of this larger field, its peculiar features, its ravines and gullies, holds the key to an appropriate understanding of any of the individual questions. Often that landscape becomes the focus of new questions.

The First Example of Perturbation Theory?



James Ferguson, *Astronomy Explained upon Sir Isaac Newton's Principles*, Project Gutenberg, 2019 (1757).

https://www.gutenberg.org/cache/epub/60619/pg60619-images.html

Problem:

Find the roots of the following quadratic equation:

$$x^2 - 4 = 0. \tag{1}$$

Solution 1: by inspection, x has roots at $x = \pm 2$.

Perturbative Solution:

Substitute the ansatz

$$x(\varepsilon) = \sum_{n \in \mathbb{N}} a_n \varepsilon^n$$

into the equation

$$x^2 - \varepsilon^p = 0.$$

We recover the original equation by taking $\varepsilon = 4^{1/p}$, i.e. $p \neq 0$.

A Simple Example

Problem:

Find the roots of the following quadratic equation:

$$x^2 - 4 = 0. \tag{1}$$

Solution 1: by inspection, x has roots at $x = \pm 2$.

Perturbative Solution:

One finds:

$$0 = \sum_{k \in \mathbb{N}} \left(\sum_{m+n=k} a_m a_n - \delta_{k,p} \right) \varepsilon^k$$

Since this holds for a variable ε we can set the coefficient of each power of ε to zero identically:

$$\sum_{m+n=k} a_m a_n = \delta_{k,p}, \qquad \forall m,\ n \in \mathbb{N}. \tag{2}$$

A Simple Example

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Solution 1: by inspection, x has roots at $x = \pm 2$.

Perturbative Solution:

Since $p \neq 0$:

- $k = 0$ requires $m = n = 0 \implies a_0^2 = 0 \implies a_0 = 0$.
- $k = 1$ has contributions $(m, n) \in \{ (1, 0), (0, 1) \}$ which vanish trivially: $p \neq 1$.
- $k = 2$: $(m, n) = (1, 1) \implies a_1^2 = \delta_{2,p} \implies a_1 = \pm 1$ if $p = 2$ and $a_1 = 0$ if $p \neq 2$.
- Recursion of above: $p = 2q$, $a_q = \pm 1$, $a_k = 0$ for $k \neq q$.

A Simple Example

Problem:

Find the roots of the following quadratic equation:

$$x^2 - 4 = 0. \tag{1}$$

Solution 1: by inspection, x has roots at $x = \pm 2$.

Perturbative Solution:

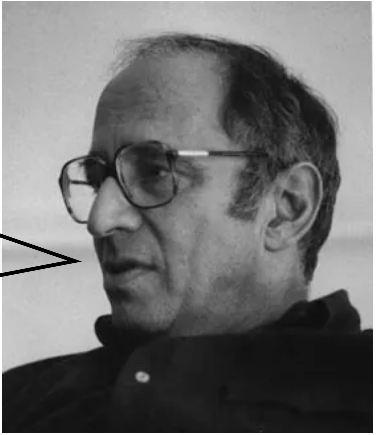
Thus we have roots at:

$$x^{\pm}(\varepsilon) = \pm \varepsilon^{q},$$

where $p = 2q$. Taking $\varepsilon = 4^{1/p}$ we find that the equation has roots at ± 2 as required.

So Why Does Perturbation Theory Work (and Why Do We Use It)?

Normal science is puzzle solving activity within a given paradigm. With improvements in experimental technique, data may become available that shows a deviation from the best known scientific model. Corrections must thus be incorporated to ensure compatibility between prevailing scientific wisdom and empirical fact.



Thomas Kuhn

https://en.namu.wiki/w/%ED%86%A0%EB%A8%B8%EC%8A%A4%20%EC%BF%A4

So Why Does Perturbation Theory Work (and Why Do We Use It)?

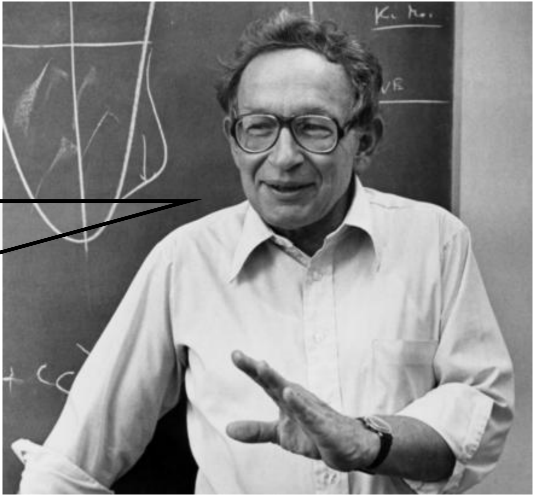
Alexander Grothendieck

https://webusers.imj-prg.fr/~leila.schneps/grothendieckcircle/photosshurik6.php

The philosophical answer is too general: here we are considering a problem where the solution lies in a space in which the polynomials are dense. For instance, for continuous functions we have by the Stone-Weierstrass theorem...

So Why Does Perturbation Theory Work (and Why Do We Use It)?

No, that is just a trivial mathematical claim! We can use perturbation theory because the situation we care about is *adiabatically connected* to a simpler scenario where the perturbative parameter is absent. Thus in Fermi liquid theory...



Philip W Anderson

Image courtesy of the AIP Emilio Segrè Visual Archives, *Physics Today* Collection

A (Slightly Less) Simple Example

Problem:

Find the roots of the following quadratic equation:

$$x^2 - 3x + 2 = 0. \tag{3}$$

Solution 1: by inspection, x has roots at $x = 1,\ 2$.

Perturbative Solution:

Substitute the ansatz

$$x(\varepsilon) := \lim_{N\to\infty} x_N(\varepsilon) \qquad x_N(\varepsilon) := \sum_{n=0}^{N} a_n \varepsilon^n$$

into the equation

$$x^2 - 3x + \varepsilon = 0.$$

A (Slightly Less) Simple Example

Problem:

Find the roots of the following quadratic equation:

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Solution 1: by inspection, x has roots at $x = 1,\ 2$.

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One finds:

$$0 = \sum_{n \in \mathbb{N}} \left(\sum_{\ell+m=n} a_\ell a_m - 3a_n + \delta_{n,1} \right) \varepsilon^n$$

Since this holds for a variable ε we can set the coefficient of each power of ε to zero identically:

$$a_n = \sum_{\ell+m=n} a_\ell a_m + \delta_{n,1}, \qquad \forall n \in \mathbb{N}. \tag{4}$$

A (Slightly Less) Simple Example

Problem:

Find the roots of the following quadratic equation:

$$x^2 - 3x + 2 = 0. \tag{3}$$

Solution 1: by inspection, x has roots at $x = 1,\ 2$.

Perturbative Solution:

- $n = 0$: $a_0^2 - 3a_0 = 0$ i.e. $a_0^{\pm} = 0,\ 3$ and $x_0(2) = 0,\ 3$.
- $a_1 = 2a_1 a_0 + 1$, i.e. $a_1^{\pm} = 1,\ -1/5$ and $x_1^{\pm}(2) = 2,\ 13/5$.
- $a_2 = 2a_0 a_2 + a_1^2$, $a_2^{\pm} = 1,\ -1/125$ and $x_2^{\pm}(2) = 6,\ 321/125$.
- ...

We have two series, one clearly not approaching any root and another apparently slowly approaching the root at $x = 2$.

A (Slightly Less) Simple Example

Problem:

Find the roots of the following quadratic equation:

$$x^2 - 3x + 2 = 0. \tag{3}$$

Solution 1: by inspection, x has roots at $x = 1,\ 2$.

Perturbative Solution:

Caution: the series typically diverges!

$$a_{n+1} \sim \sum_{\ell+M=n} a_\ell a_m \sim \mathcal{O}(1)(n+1)a_0 a_n \qquad a_n \sim \mathcal{O}(1)a_0^n n! \tag{5}$$

i.e.

$$x_N(\varepsilon) \sim \exp\left(N \log\left(\frac{N}{a_0\varepsilon}\right)\right). \tag{6}$$

Basic Problems of Perturbation Theory

- What is it about nature (or physics) that makes perturbation theory useful?
    - Why is nature not exactly solvable/integrable?
    - How do distinct scales ‘decouple’ in renormalisation theory.
- What is the significance of polynomials for perturbation theory?
    - Regularity of physical relations.
- Why does perturbation theory diverge?
- Why does perturbation theory continue to be useful despite diverging?
    - Do we still have a connection principle like adiabatic continuity for divergent perturbation theory?
- Why is perturbation theory even more useful than it should be?
    - Resummation techniques show that perturbative expansions encode more information than originally expected.

Part II

Solvability, Integrability and their Limits

Some Caveats (To Be Ignored in the Subsequent)

- Solvable $\neq$ Integrable.
- Integrable $\neq$ Liouville integrable.
- Quantum $\neq$ Quantised.
- Quantisation $\neq$ Deformation quantisation.

Hitchin's Characterisation of Integrability

Nigel Hitchin, *Introduction* to *Integrable Systems: Twistors, Loop Groups and Riemann Surfaces* by Hitchin, Segal and Ward

Integrable systems, what are they? It's not easy to answer precisely. The question can occupy a whole book … or be dismissed as Louis Armstrong is reputed to have done once when asked what jazz was—'If you gotta ask, you'll never know!' … If we steer a course between these two extremes, we can say that integrability of a system of differential equations should manifest itself through some generally recognizable features:

- the existence of many conserved quantities;
- the presence of algebraic geometry;
- the ability to give explicit solutions.

These guidelines should be interpreted in a very broad sense: the algebraic geometry is often transcendental in nature, and explicitness *doesn't* mean solvability in terms of sines, exponentials or rational funvtions.

Definition

Let $(\mathcal{M}, \omega)$ be a symplectic $2n$-manifold—or more generally let $(\mathcal{M}, \{\cdot, \cdot\})$ be a Poisson manifold (phase space). A *(Liouville) integrable system* is an algebra $\mathcal{A} \subseteq \mathcal{C}^{\infty}(\mathcal{M})$ such that $\dim(\mathcal{A}) = n$ and

$$\{f, g\} = 0 \tag{7}$$

for all f, $g \in \mathcal{C}$.

Remark

The algebra $\mathcal{A}$ is generated by n linearly independent constants of motion $f_1 = H, f_2, ..., f_n$. By linearity and anticommutativity of the Poisson bracket, the integrability condition 7 follows from the requirement that the constants of motion are in involution:

$$\{f_k, f_\ell\} = 0 \qquad \forall k,\ \ell = 1, ..., n. \tag{8}$$

Theorem (Liouville-Arnold)

Let $\mathcal{A}$ be an integrable system in $(\mathcal{M}, \omega)$ generated by $H, f_1, \ldots, f_{n-1}$ and let $(q,p) \in \mathcal{M}$ be a regular point for the mapping

$$F = (F, f_1, \ldots, f_n) : \mathcal{M} \to \mathbb{R}^n \qquad F : (q,p) \mapsto (H(q,p), f_1(q,p), \ldots, f_{n-1}(q,p)). \quad (9)$$

Let $\mathcal{T}(q,p) = F^{-1}(F(q,p))$ denote the level-set of F at $F(q,p)$.

1. *$\mathcal{T}(q,p)$ is an imbedded n-dim. submanifold of $\mathcal{M}$ invariant under the phase flow.*
2. *If the motion is bounded and of a single system ($\mathcal{T}(q,p)$ is compact and connected) then $\mathcal{T}(q,p) \cong \mathbb{T}^n$ i.e. $\mathcal{T}(q,p)$ is diffeomorphic to an invariant n-torus.*
3. *There exist action-angle coordinates (ω, ϕ) for $\mathcal{M}$ such that*

$$\mathcal{T}(q,p) \cong \{\, \phi \bmod 2\pi \,\} \qquad \dot{\phi} = \omega \qquad \omega = \omega(F). \quad (10)$$

In particular ω is constant on $\mathcal{T}(q,p)$.

Sketch of Proof of the Liouville-Arnold Theorem I

- (1) follows from the inverse (implicit) function theorem.
- We can construct n linearly independent vector fields

$$v_k = (\nabla_{\tilde{p}} f_k, -\nabla_{\tilde{q}} f_k) \tag{11}$$

(where $f_0 := H$).

- Since $\mathcal{T}(q,p)$ is defined by $\nabla_{\tilde{q}} f_k = \nabla_{\tilde{p}} f_k = 0$, the normals to $\mathcal{T}(q,p)$ are the vector fields $(\nabla_{\tilde{q}} f_k, \nabla_{\tilde{p}} f_k)$.
- But by integrability

$$v_k \cdot (\nabla_{\tilde{q}} f_\ell, \nabla_{\tilde{p}} f_\ell) = \nabla_{\tilde{p}} f_k \nabla_{\tilde{q}} f_\ell - \nabla_{\tilde{q}} f_k \nabla_{\tilde{p}} f_\ell = \{f_k, f_\ell\} = 0 \tag{12}$$

and the vector fields v_k form a tangent basis at each point.

Sketch of Proof of the Liouville-Arnold Theorem II

- Each vector field v_k defines an action of $\mathbb{R}$ on $\mathcal{T}(q, p)$ that generates a natural flow. $\mathcal{T}(q, p)$ is clearly invariant under the flow (since the flow is parallel to $\mathcal{T}(q, p)$ you do not escape $\mathcal{T}(q, p)$ by following the flow).
- Also since $\mathcal{T}(q, p)$ is compact, flowing along v_k will eventually lead to a return to your original position. If τ_k is the initial return time then the little group at any point along k is given by $\{ n\tau_k : n \in \mathbb{Z} \} \subseteq \mathbb{R}$.
- $\mathcal{T}(q, p)$ is thus naturally seen as a quotient of $\mathbb{R}^n$ by a discrete subgroup and hence a torus.

Phase Flow of Points on Invariant Tori

- Each point exhibits a natural phase flow on the invariant torus under the tangent vector fields.
- In action angle variable $\dot{\phi} = \omega$, i.e. we have points circulating the independent circles of the torus with constant frequency.
- If the frequencies are commensurate a point will eventually return to where it began under the phase flow.
- Otherwise the trajectory will be *quasiperiodic*, densely covering the invariant torus but never returning to its start point.

Definition

Let (H_0, ω, ϕ) be an integrable system, i.e. $H_0 = H_0(\omega)$, $\dot{\phi} = \omega$. A *perturbation* to the integrable system (H_0, ω, ϕ) is a Hamiltonian system (H, ω, ϕ) such that

$$H = H_0(\omega) + \varepsilon H_1(\omega, \phi) + \cdots \ . \tag{13}$$

A *solution* to the perturbed system (H, ω, ϕ) consists of a triple $(S, \tilde{\omega}, \tilde{\phi})$ with the following properties:

1. $(\tilde{\omega}, \tilde{\phi})$ are new coordinates such that $H(\omega, \phi) = \tilde{H}(\tilde{\omega}, \varepsilon)$ for some $\tilde{H}$.
2. S is an analytic function of ε satisfying the following properties:

$$S = S(\tilde{\omega}, \phi, \varepsilon) = \sum_{n \in \mathbb{N}} S_n \varepsilon^n \qquad \omega = -\frac{\partial S}{\partial \phi} \qquad \tilde{\phi} = -\frac{\partial S}{\partial \phi} \tag{14}$$

where $S_0 = \tilde{\omega}\phi$ and the functions $S_k(\tilde{\omega}, \phi)$ are periodic in ϕ.

Remark

With this definition S is the generator of a canonical transformation into new action-angle variables for the perturbed Hamiltonian system.

- We can try to find a solution to the perturbed system H.
- First note that a solution satisfies the following first-order Hamilton-Jacobi equation:

$$\nabla_{\tilde{\omega}} H_0 \cdot \nabla_{\phi} S_1 + H_1(\tilde{\omega}, \phi) = \tilde{H}_1(\tilde{\omega}) \tag{13}$$

for some function $\tilde{H}_1$ that remains to be determined.
- Since H_1 is periodic in ϕ we have the Fourier expansion

$$H_1 = \sum_{m \in \mathbb{Z}^n} H_m(\tilde{\omega}) \exp(im \cdot \phi). \tag{14}$$

If $\tilde{H}_1$ is periodic we then have

$$\tilde{H}_1(\tilde{\omega}) = \frac{1}{(2\pi)^n} \int_{\mathbb{T}^n} \mathrm{d}^n \phi H_1(\tilde{\omega}, \phi). \tag{15}$$

- Expanding

$$S_1 = \sum_{m \neq 0} S_{1,m}(\tilde{\omega}) \exp(im \cdot \phi) \tag{16}$$

gives

$$S_{1,m}(\tilde{\omega}) = \frac{H_m(\tilde{\omega})}{im \cdot \omega(\tilde{\omega})}. \tag{17}$$

- ω has a *resonance* at m if:

$$m \cdot \omega = 0. \tag{18}$$

Clearly $S_{1,m}$ is not defined if ω has a resonance at m. S_1 is not defined if we have sufficiently many m such that ω is *near* resonance at m.

- This is the problem of *small divisors*.

Definition

- A vector ω is *resonant* iff there is an $m \in \mathbb{Z}^n \setminus \{0\}$ such that
$$m \cdot \omega = 0. \tag{19}$$
Otherwise ω is *nonresonant.*
- ω is *strongly nonresonant* iff there are constants α, $\beta > 0$ such that
$$m \cdot \omega \geq \frac{\alpha}{|k|^{\beta}} \tag{20}$$
for all $k \in \mathbb{Z}^n \setminus \{0\}$.

KAM Theorem

Resonances are both *generic* but *rare*. Strong nonresonances are *non-generic* but *common*.

Fact

- *The resonant vectors are a dense set of measure zero.*
- *The strongly resonant vectors are a nowhere dense set of full measure.*

KAM Theorem

Theorem (KAM)

Let $H = H(\varepsilon)$ be a perturbation of an integrable Hamiltonian system, let $\alpha \in (0, \infty)$ be sufficiently small and fix $\beta > n - 1$. There is a constant $\delta > 0$ such that for any fixed $\varepsilon \in (0, \infty)$ with

$$\varepsilon < \delta\alpha^2 \tag{19}$$

the perturbed Hamiltonian system admits a solution defined locally on $U \times \mathbb{T}^n$. Every invariant torus of the unperturbed system with strongly nonresonant frequencies survives the perturbation and the set of all such tori fills $U \times \mathbb{T}^n$ up to a set of measure $\mathcal{O}(\alpha)$.

A (the?) fundamental assumption in classical statistical mechanics:

Ergodic Hypothesis

Ensemble averages can be computed from time averages in the microcanonical ensemble because in the limit of large time a typical classical system traverses all of phase space in such a way that the time spent in a given phase region is determined by the volume of that region.

Definition

Consider a dynamical system (Ω, μ, ϕ_t) where $f_t : \Omega \to \Omega$ describes the (Hamiltonian) flow of the system. The system is *ergodic* iff for all non-null measurable sets $E \subseteq \Omega$ and μ-almost all $\omega \in \Omega$ there is a $t \in \mathbb{R}$ such that $\phi_t(\omega) \in E$.

Remark

The above definition essentially states that in an ergodic system you always travel to every region of phase space under the natural phase flow.

Theorem (Ergodic Theorem)

Define the phase and time averages of a function f as

$$\mathbb{E}(f) := \frac{1}{\mu(\Omega)} \int_{\Omega} \mathrm{d}\mu(\omega) f(\omega) \qquad \bar{f}(\omega_0) := \lim_{T\to\infty} \frac{1}{T} \int_0^T \mathrm{d}t f(\phi_t(\omega_0)) \tag{20}$$

respectively. Then the following statements are equivalent:

1. *For μ-almost all initial conditions $\omega_0 \in \Omega$ and any integrable phase function f, the phase average is equal to the time average:*
$$\mathbb{E}(f) = \bar{f}(\omega_0) \tag{21}$$
2. *(Ω, μ, ϕ_t) is ergodic.*

Corollary

The microcanonical ensemble is the unique distribution on a constant energy surface in the canonical phase space $\mathbb{R}^{2n}$ that is invariant under the natural Hamiltonian flow and absolutely continuous with respect to the Lebesgue measure.

Integrable Systems are Not Ergodic!

- Integrable systems—confined to an n-dimensional invariant torus in a $2n$-dimensional phase space—are obviously not ergodic since they cannot traverse the remaining $(n-1)$-dimensions of the constant energy surface.
- The KAM theorem suggests that approximate ergodicity cannot be restored for integrable systems through the fact that we do not have an exact knowledge of the microscopic dynamics.

Some Caveats

- These facts are often seen as defects with the ergodic foundations of statistical mechanics rather than the as arguments for the rarity of integrable systems.
- Proving that dynamical systems are ergodic is typically very hard and has not been achieved for even the simplest nontrivial examples in statistical mechanics.

- Nontrivial topology of configuration space.
- Splitting of asymptotic manifolds.
- Branching of solutions after analytic continuation.
- ‘Quasi-random’ oscillations.
- …

Remark

The first obstruction is *kinematic*. The remaining obstructions are dynamical obstructions relating to *resonance phenomena* that destroy invariant tori.

C.f. Kozlov, *Symmetries, Topology and Resonances in Hamiltonian Mechanics*.

Theorem

Consider a Hamiltonian system with two bounded degrees of freedom, described by the compact space $\mathcal{M}$. If $\chi(\mathcal{M}) < 0$, every analytic first integral on phase space $T^*\mathcal{M}$ depends on the energy.

Lemma

Let $\Sigma_E \subseteq T^*\mathcal{M}$ be the constant energy surface for a Hamiltonian $H = T + V$ such that

$$E < \sup_{q \in \mathcal{M}} V(q). \tag{20}$$

Then every analytic integral of the motion is a constant function on Σ_E.

- By the Jacobi-Maupertius principle, the trajectories of the mechanical system with $E > V$ in $\mathcal{M}$ are geodesics with respect to the *Jacobi metric*

$$ds_J^2 = 2(E - V)ds^2 \qquad T = \frac{1}{2}\left(\frac{ds}{dt}\right)^2 . \tag{21}$$

- By the Gauss-Bonnet formula

$$\chi(\mathcal{M}) = \frac{1}{2\pi}\int_{\mathcal{M}} d\,\mathrm{vol}_{\mathcal{M}}(q) R(q) \tag{22}$$

where $R(q)$ is the Ricci scalar of the Jacobi metric.

- If $\chi(\mathcal{M}) < 0$ then the system is "hyperbolic on average".
- If $\mathcal{M}$ is hyperbolic then the phase flow is described by *Anosov diffeomorphisms*.
- Such an *Anosov system* is ergodic on Σ_E.

A Conflict?

- We have asserted that most classical systems are not integrable on the grounds of resonances that spoil integrability.
- On the other hand we have seen that such resonances are sufficiently rare that integrability is typically preserved under perturbations due to the KAM theorem.
- The modern picture is that the rare resonances nonetheless give rise to complex trajectories in phase space with apparently irregular *effectively stochastic* motions.
- These motions are not negligible because they are dense in phase space and occur with nonzero (if small) probability under perturbation.
- Actual motion in phase space given the presence of random perturbations is thus complex mixture of regular motion on invariant tori and essentially stochastic ergodic trajectories.

Quantum Integrability

Definition

A *quantum integrable system* for a quantum system with n-degrees of freedom is a family of n independent operators that are functions of the positions and momenta and which mutually commute.

Remark

More generally a quantum integrable system is a ‘maximal’ commutative subalgebra of a von Neumann algebra where the notion of ‘maximal’ depends on the context.

Definition

Let $\mathcal{A}$ be an algebra. $\mathcal{A}[[x]]$ is then the algebra of formal power series $\alpha := \sum_{n\in\mathbb{N}} \alpha_n x^n$ where $\alpha_n \in \mathcal{A}$ for all $n \in \mathbb{N}$. Addition and scalar multiplication ared defined elementwise and a product is given by

$$\alpha\beta = \sum_{n\in\mathbb{N}} \left(\sum_{k\leq n} \alpha_k \beta_{n-k} \right) x^n. \tag{23}$$

Remark

We have a natural imbedding

$$\mathcal{A} \subseteq \mathcal{A}[[x]] \qquad \alpha \mapsto \alpha + 0x + 0x^2 + \cdots . \tag{24}$$

Definition

Let $(\mathcal{A}, \{\cdot, \cdot\})$ be a Poisson algebra, i.e. a Lie algebra such that the Lie bracket satisfies the following Leibniz product law:

$$\{f, gh\} = \{f, g\}h + g\{f, h\}. \tag{25}$$

A *deformation quantisation* of $(\mathcal{A}, \{\cdot, \cdot\})$ is an associative unital product $\star$ on $\mathcal{A}[[\hbar]]$ such that

$$\alpha \star \beta = \alpha\beta + \mathcal{O}(\hbar) \tag{26}$$

and

$$[\alpha, \beta] := \alpha \star \beta - \beta \star \alpha = i\hbar\{\alpha, \beta\} + \mathcal{O}(\hbar^2). \tag{27}$$

- Deformation quantisation defines a quantum system as a perturbative expansion around a classical system in powers of $\hbar$.
- Thus by the KAM theorem we expect that *most* deformation quantisations of a classical integrable system will determine a quantum integrable system.
- Robnik[1] argues that this claim can be improved: he conjectures that all (discrete) quantum systems are strongly nonresonant with a constant $\alpha = \alpha(\hbar)$. Classical resonances arise in the $\hbar \to 0$ when $\alpha(\hbar)$ diverges.
- In short: *quantum fluctuations lift classical resonances*.

[1] Marko Robnik, 'A note on quantum integrability', Journal of Physics A: Mathematical and General, 19, L841–L847, 1986. doi: 10.1088/0305-4470/19/14/004.

Definition

Let $(\mathcal{A}, \{\cdot,\cdot\})$ be a Poisson algebra and $\mathcal{C} \subseteq \mathcal{A}$ an integrable system in $\mathcal{A}$. We say that $\mathcal{C}$ determines a *quantum integrable system* iff $\mathcal{C}[[\hbar]]$ is a commutative subalgebra of $(\mathcal{A}[[\hbar]], \star)$, a deformation quantisation of $(\mathcal{A}, \{\cdot,\cdot\})$.

Theorem (Garay-van Straten)

Let $\mathcal{M}$ be a symplectic manifold (phase space) and consider the Poisson algebra $(\mathcal{C}^\infty(\mathcal{M}), \{\cdot,\cdot\})$. An integrable system $\mathcal{C} \subseteq \mathcal{C}^\infty(\mathcal{M})$ determines a quantum integrable system, i.e. there is a deformation quantisation $(\mathcal{C}^\infty(\mathcal{M})[[\hbar]], \star)$ of $(\mathcal{C}^\infty(\mathcal{M}), \{\cdot,\cdot\})$ such that $\mathcal{C}[[\hbar]]$ is commutative.

Korepin, Bogoliubov and Izergin, *Quantum Inverse Scattering Method and Correlation Functions*

Bethe Ansatz solvable models are not free; they generalize free models of quantum field theory in the following sense. Many-body dynamics of free models can be reduced to one-body dynamics. With the Bethe Ansatz, many-body dynamics can be reduced to two-body dynamics. The many-particle scattering matrix is equal to the product of two-particle ones. This leads to the self-consistency relation for the two-particle scattering matrix. It is the famous Yang-Baxter equation ... which is the central concept of exactly solvable models.

Most Quantum Systems are Not Integrable(?)

Argument 1:

- If a quantum system is in a thermal pure state it is widely expected to obey the *Eigenstate Thermalisation Hypothesis*.
- Quantum integrable systems often fail to obey ETH.

Argument 2:

- A quantum integrable system is one in which many body dynamics is totally determined by two-body dynamics.
- In large systems we expect significant differences in the dynamics due to emergent n-body effects: n-particle interactions (scattering amplitudes) do not factorise into two-body interactions, multipartite entanglement...

Argument 3: commutative subalgebras of von Neumann algebras are rare (?)

Task: Prove the existence of a solution to some differential equation

$$Df = 0. \tag{28}$$

Procedure:

1. Recast the problem as an optimisation problem: f solves 28 iff it solves

$$f = \inf_{g \in V} \mathcal{A}(g). \tag{29}$$

   E.g. equation 28 is an Euler-Lagrange equation for some action $\mathcal{A}$.
2. Show that we can compute the infimum for acompact feasible set V.
3. Show that $\mathcal{A}$ is (lower semi)continuous.
4. Apply the following fact:

Fact

Every lower semicontinuous on a compact space attains its minimum.

- The procedure sketched previously is completely *nonconstructive*.
- This follows since it relies on a general fact that has nothing to do with the problem in question.
- Even worse this general fact relies on a notion—*compactness*—whose utility depends on the existence of different equivalent formulations.
- One of those formulations makes the existence of a minimiser obvious (in fact is essentially equivalent to the existence of a minimiser).
- The standard formulation of compactness, however, does not make this minimisation statement apparent, but may be used to show more flexibly that the feasible set is compact.
- The equivalence of these distinct formulations relies on (some variant of?) the *axiom of choice*, i.e. the utility of this general theorem is *fundamentally nonconstructive*.

- Finding action-angle variables in general involves finding the inverse of a nonlinear equation.
- Such functions may be very hard to compute even if we know that a solution exists.

Example

Find the inverse to the function

$$f(x) = x + \cos x \qquad x \in \left[0, \frac{\pi}{2}\right). \tag{30}$$

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$$g_0(x) = x \qquad g_{n+1}(x) = x - \cos(g_n(x)) \qquad g(x) = \lim_{n \in \mathbb{N}} g_n(x). \tag{31}$$

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- i.e.

$$g_3(x) = x - \cos(x - \cos(x - \cos(x))).$$

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- Thus rearranging we have

$$x = g(x) + \cos(g(x)) = f(g(x)).$$

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- Similarly, $g(f(x)) = f(x) - \cos(g(f(x)))$ i.e.

$$f(x) = f(g(f(x))).$$

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- $f'(x) = 1 - \sin(x)$ i.e. $f'(x) > 0$ on the domain. By the implicit function theorem f is thus injective and so $f(x) = f(g(f(x)))$ implies that $g(f(x)) = x$.

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- It only remains to prove that $g(x)$ exists. Exercise!

Definition

An injective computable function $f : \mathbb{N} \to \mathbb{N}$ is *computably invertible* iff there is a computable partial function $g : \mathbb{N} \to \mathbb{N}$ such that

$$g(x) = \begin{cases} y, & x = f(y) \\ \perp, & x \notin f(\mathbb{N} \end{cases} \tag{32}$$

Fact

If f is computably invertible then the range of f is computable i.e. there is a finite time algorithm that correctly identifies the elements of the set.

Proof.

The function

$$h(x) = \begin{cases} 1, & g(x) \neq \perp \\ 0, & g(x) = \perp \end{cases}$$

identifies the range of f and is computable if g is computable. □

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Fact

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Counterexample

There exist injective computable functions with noncomputable ranges.

Definition

A partial function $f : \mathbb{N} \to \mathbb{N}$ is called *one way* iff it is

1. Injective.
2. *Honest:* there is a polynomial p such that for each $y \in \mathrm{cod}(f)$ there is an $x \in f^{-1}(y)$ such that $|x| \leq p(|y|)$.
3. Can be computed by a deterministic polynomial time algorithm.
4. f^{-1} cannot be computed with a deterministic polynomial time algorithm.

Definition

The complexity class **UP** consists of decision problems that admit an *unambiguous non-deterministic polynomial time algorithm*. In particular a decision problem belongs to **UP** iff it can be solved in polynomial time on a non-deterministic Turing with at most one accepting path for each input.

Fact

- $\mathbf{P} \subseteq \mathbf{UP} \subseteq \mathbf{NP}$.
- $\mathbf{P} \stackrel{?}{=} \mathbf{UP}$ *remains an important open problem.*
- $\mathbf{P} \neq \mathbf{UP}$ *implies* $\mathbf{P} \neq \mathbf{NP}$.
- *A one-way function exists iff* $\mathbf{P} \neq \mathbf{UP}$ *(Grollman-Selman-Ko).*

Florin Diacu, 'The Solution to the N-Body Problem', *Mathematical Intelligencer*, 18, 66-70, 1996.

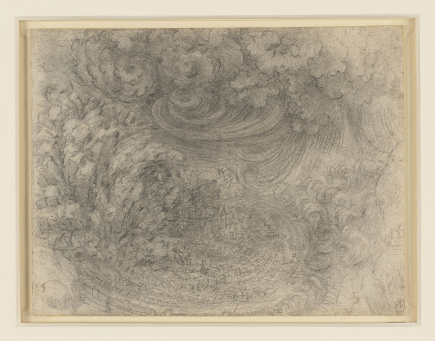
In 1991, a Chinese student, Quidong (Don) Wang, published a beautiful paper in which he provided a convergent power series solution to the *n*-body problem. ... But did this mean the end of the *n*-body problem? Though he provided a solution as defined in sophomore textbooks, does this imply we know everything about gravitating bodies, about the motion of stars and planets in the universe? Paradoxically, we do not, in fact we know nothing more than before having the solution. ... [T]hese series solutions, though convergent on the whole real axis, are practically useless because of their very slow rate of convergence. In applications, one would have to sum up millions of terms to determine the motion of particles for insignificantly short intervals of time. This unusual situation ... clarifies that even a constructive solution can be useless from a practical point of view.

Reductively Fundamental Theories Might Be Integrable

- String theory is perhaps the TOE and it is integrable.
- Every UV/IR fixed point of a renormalisation group flow is a conformal field theory—these are integrable (in a slightly wider sense than the above integrable systems) due to the structure of the conformal bootstrap: all information is reduced to the nature of the three-point structure constants.
- Most physical systems of interest are not reductively fundamental; instead they concern relatively isolated phenomena at certain characteristic (energy) scales.
- In particular a good physical system is concerned with phenomena that decouple from physical effects occurring at distinct scales.

The Nature of Physical Decoupling

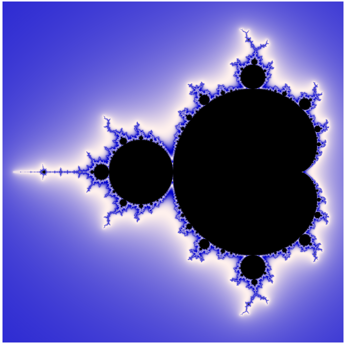
- The precise meaning of the decoupling of physical effects at different scales is relatively subtle.
- In particular it does not mean that physics at one scale is independent of physics at another scale.



Turbulence, depicted in Da Vinci's *A Deluge*

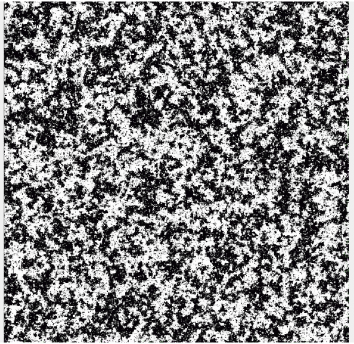
RL 12378

https://www.rct.uk/collection/912378/a-deluge



Fractals

https://commons.wikimedia.org/wiki/File:Mandelbrot20210909_ABC02_65535x65535.png



Statistical models at criticality

https://commons.wikimedia.org/wiki/File:Ising_Criticality2.gif

- Compute the overlap of two quantum states $|\psi\rangle$ and $|\chi\rangle$.
- Expand each in an energy eigenbasis $\{\ |\ e_n\rangle\ \}$:

$$|\psi\rangle = \sum_n \psi_n \,|e_n\rangle \qquad\qquad |\chi\rangle = \sum_n \chi_n \,|e_n\rangle\,. \tag{33}$$

- Then

$$\langle\psi|\phi\rangle = \sum_n \psi_n^* \phi_n. \tag{34}$$

- The sum over n is a sum over all the distinct energy levels of the system:
- Quantum phenomena generically have characteristic contributions at all scales.

- The partition function for a quantum particle of mass m moving on the line is given

$$\mathcal{Z}[J] = \int_{\Gamma} \mathcal{D}\gamma \exp\left(\frac{i}{2} m \int_{\gamma} \d{t} \dot{\gamma}^2(t)\right) \tag{33}$$

where Γ is a suitable space of paths.

- Thus trajectories of all possible values of the action (and thus energy) contribute to the quantum partition function.
- Expectations can be computed via functional derivatives of the partition function.
- Thus quantum expectations incorporate contributions from possible trajectories at all scales.

- The above arguments suggest that quantum fluctuations generically appear at all energy scales.
- Further evidence of this is the appearance of conformal (Weyl) anomalies: quantum fluctuations typically break classical scale invariance.
- Naturalness problems (hierarchy problem, cosmological constant problem) concern the sensitivity of measurable quantities to phenomena at very distinct scales.

- The effect of quantum fluctuations at energy scales much higher than at present concern can be integrated out.
- If an interaction that takes place at a higher energy scale is not characteristic of the present energy scale then it disappears from the effective action describing the theory at the current energy scale.
- However it leaves a residue of its presence by *shifting the value of the coupling constants of the theory*.
- The renormalisation group encodes how the coupling constants of a theory flow in parameter space.
- Since quantum fluctuations generically induce all possible interactions, the parameter space is an infinite dimensional space of all possible couplings.
- The flow in this infinite dimensional space can thus be encoded in a *Wilsonian effective field theory* including all possible interaction terms.

- Applequist and Carrazone introduced a *decoupling theorem* which precisely characterises the way low-energy physics 'decouples' from high-energy physics.
- Assume we have a light scalar φ and a heavy scalar Φ of masses m and M respectively interacting via the term

$$\lambda\phi^2\Phi^2. \tag{34}$$

- Let Σ_Λ denote the 1-loop scalar two-point function with cutoff energy Λ i.e.

$$\Sigma_\Lambda = m^2(\Lambda) + \lambda(\Lambda)\int_{|k|<\Lambda}\frac{d^4k}{(2\pi)^4}\frac{1}{k^2+M^2(\Lambda)}. \tag{35}$$

- Since this is a physical quantity it should not depend on the cutoff:

$$0 = \Sigma_{\Lambda_0} - \Sigma_\Lambda \tag{36}$$

where we take $\Lambda_0 \gg M$ to be a high-energy cutoff scale and Λ our present energy scale.

- Ignoring subleading terms the above implies

$$m^2(\Lambda) = m^2(\Lambda_0) + \frac{\lambda(\Lambda_0)}{16\pi^2}\left((\Lambda_0 - \Lambda) + M^2(\Lambda_0)\ln\frac{\Lambda_0^2 + M^2(\Lambda_0)}{\Lambda^2 + M^2(\Lambda)}\right) + \mathcal{O}(\lambda^2). \quad (34)$$

- In the low energy limit $\Lambda \ll M$ we find

$$m^2(\Lambda) \sim M^2(\Lambda_0)\ln\left(1 + \frac{\Lambda_0^2}{M^2(\Lambda_0)}\right). \quad (35)$$

- Clearly the value of $m^2(\Lambda)$ depends on both the heavy mass M and the UV cutoff scale Λ_0.

- For $\Lambda \ll M$ we see that $M^2(\Lambda_0)$ contributes only an *additive shift* to the value of $m^2(\Lambda)$.
- Thus for a measurement of $m^2(\Lambda)$ at a *single* value of Λ one cannot separate the contributions from m and M at Λ_0.
- We thus need to study how $m^2(\Lambda)$ varies with Λ: we want the RG equation.
- Differentiating the above gives

$$\frac{\mathrm{d}m^2(\Lambda)}{\mathrm{d}\ln\Lambda} = \frac{\lambda(\Lambda_0)}{8\pi^2}\Lambda^2\left(\frac{M^2(\Lambda_0)}{\Lambda^2 + M^2(\Lambda)} - 1\right) \tag{34}$$

  where we assume that M varies slowly with Λ compared to m.
- With this assumption we have $M^2(\Lambda_0) \sim M^2(\Lambda)$ and at $\Lambda \ll M$ we find

$$\frac{\mathrm{d}m^2(\Lambda)}{\mathrm{d}\ln\Lambda} \neq f(M). \tag{35}$$

Applequist-Carrazone Theorem

- The decoupling theorem shows us that at low energies the heavy-mass scalar decouples from the light-mass scalar in the sense that it does not contribute to the *running* of the light-mass.
- Thus at light masses the renormalisation group trajectory is confined to an *affine subspace* of the (m, M) plane.
- The actual physics (values of m and M) still depends on the *UV* scale through the initial condition for the masses m etc. i.e. in the choice of affine subspace.
- Since $m(\Lambda_0)$ and $M^2(\Lambda_0)$ do not change as we scale Λ and since these terms contribute additively to the value of m for small Λ we see that doubling Λ does not lead to doubling m.
- Thus being confined to a nontrivial affine subspace (rather than a linear subspace) is associated with a quantity having an *anomalous dimension*.

I borrow a kettle from my neighbour. My neighbour claims the kettle is broken and asks me to buy them a new one. I refuse because:

1. I returned the kettle undamaged.
2. The kettle was already damaged when I borrowed it.
3. I never borrowed the damn kettle anyway!

Trustees of the British Museum.
https://www.britishmuseum.org/collection/image/1205873001

My neighbour gives me some equations of motion and asks me to solve them. I refuse because:

1. The equations of motion do not have a solution (since the system described by the equations of motion is not integrable).
2. They have a solution but it isn't useful (since it is nonconstructive, too implicit or not efficiently computable).
3. They aren't the right equations of motion anyway!

Part III

Regularity of Physical Relations

Précis of the Argument

- There is a mathematical distinction between *analyticity* and *smoothness*. In particular, while all analytic functions are smooth, typical smooth functions are not analytic.
- We do not expect the functional relations of physics to be described by analytic functions *unless there is a good external explanation for analyticity.*
- Smooth functions are characterised by *approximate polynomiality*.
- Polynomials in turn are characterised by the fact they characterise composite systems exhibiting (integer) power-law scaling around variable centres.
- The theory of regularity structures is a general framework for studying the expansion of singular distributions in terms of other distributions, vastly generalising the Taylor expansion.

Definition

Consider a mapping $f : \mathbb{R}^n \to \mathbb{R}^D$.

- f is *continuous* iff for every $x \in \mathbb{R}^n$ and every $\varepsilon > 0$ there is a $\delta > 0$ such that

$$||f(x) - f(y)|| < \varepsilon$$

  for all $y \in \mathbb{R}^n$ such that $||x - y|| < \delta$.

- f is *differentiable* iff there is a *linear* map $\Lambda : \mathbb{R}^n \to \mathbb{R}^D$ such that

$$\lim_{\delta x \to 0} \frac{||f(x + \delta x) - f(x) - \Lambda x||}{||\delta x||} = 0 \tag{34}$$

- f is C^k iff it is k-times differentiable and the kth derivative is continuous. f is *smooth* or C^∞ iff it is C^k for all $k \in \mathbb{N}$.

Remark

- Intuitively a continuous function is one that *has no gaps*, while a smooth (differentiable) function is one that looks *locally linear*.
- Every smooth function is continuous but there are many counterexamples to the converse claim. For instance take a typical trajectory of Brownian motion.

Theorem

Let $f : \mathbb{R} \to \mathbb{R}$ be C^k at the point $a \in \mathbb{R}$. Then there is a function $\varepsilon_k : \mathbb{R} \to \mathbb{R}$ such that

$$\lim_{x \to a} \varepsilon_k(x) = 0 \tag{34}$$

and

$$f(x) = \sum_{n=0}^{k} \frac{1}{n!} f^n(a)(x-a)^n + \varepsilon_k(x)(x-a)^k. \tag{35}$$

Warning

Taylor's theorem says nothing about the convergence of the infinite Taylor series, even for smooth functions.

Definition

A function $f : \mathbb{R} \to \mathbb{R}$ is *analytic* iff for each $a \in \mathbb{R}$ it can be described by a convergent power series:

$$f(x) = \sum_{n \in \mathbb{N}} a_n (x-a)^n. \tag{36}$$

Corollary

For an analytic function $f(x) = \sum_{n \in \mathbb{N}} a_n (x-a)^n$ we have

$$a_n = \frac{1}{n!} f^n(a) \qquad f(x) = \sum_{n \in \mathbb{N}} \frac{1}{n!} f^n(a)(x-a)^n. \tag{37}$$

Moreover the Taylor series can be expanded around any point of the analytic domain.

Lemma (Fundamental Lemma)

Suppose that a function f is analytic in some interval (a, b) and assume that its derivatives all vanish at some point $c \in (a, b)$. Then f is constant on the interval (a, b).

Proof.

Since f is analytic we have $f(x) = \sum_n \frac{1}{n!} f^n(c)(x-c)^n$ for all $x \in (a, b)$. But since $f^n(c) = 0$ for all $n > 0$ we have $f(x) = f^0(c) = f(c)$ for all $x \in (a, b)$ as required. □

Example

The function

$$f(x) = \exp\left(-\frac{1}{x^2}\right) \tag{38}$$

has a convergent Taylor series that is distinct from f.

Proof.

Clearly

$$f^n(x) = \frac{P_n(x)}{x^{2n}} \exp\left(-\frac{1}{x^2}\right)$$

where P_n is a polynomial. The x^{-2n} factor leads to a possible singularity at $x = 0$ but this is regulated by the exponential term. Thus f^n exists and is continuous on the entire domain and f is C^∞. By the same argument we see that $f^n(0) = 0$ and the Taylor series vanishes identically by the fundamental lemma. □

Lemma

Let $f, g : \mathbb{R} \to \mathbb{R}$ be analytic in some interval (a, b) and suppose that $f = g$ when restricted to the smaller interval (c, d) where $a \leq c < d \leq b$. (Of course the case $a = c$, $b = d$ is trivial so in general we assume that equality holds for at most one of these identities holds.) Then $f = g$ on (a, b).

Proof.

Since $f = g$ on (c, d), $f^n(\alpha) = g^n(\alpha)$ for each $\alpha \in (c, d)$. But then $(f - g)^n(\alpha) = 0$ for all n and $(f - g)$ is constant on (a, b) by the fundamental lemma. Since $(f - g)(\alpha = 0$ we thus have $f = g$ on (a, b) as required. □

Theorem

Let f be analytic and nonzero in some interval (a, b). Then the support of f must contain (a, b). (Recall that the support of a function f is the closure of the set of points for which $f(x) \neq 0$.) In particular there is no analytic function of compact support.

Proof.

Let f be an analytic function and let K denote the support of f. Since f is nonzero on (a, b), $K \cap (a, b) \neq \emptyset$.Then since K is closed, $K^c := \mathbb{R} \backslash K$ is open and either $(a, b) \subseteq K$ or $K^c \cap (a, b)$ is nonvoid and contains some interval $(c, d) \subset (a, b)$. But in the latter case, $f = 0$ on (c, d) and so $f = 0$ on (a, b) by analytic continuation. □

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Analytic Perturbations

The above result shows that analytic functions are very rigid. Indeed there are no local analytic perturbations: an analytic function must modify another function everywhere in its analytic domain.

Example

The following is an example of a smooth function with compact support

$$f(x) = \begin{cases} \exp\left(-\frac{1}{1-x^2}\right), & x \in [-1,1] \\ 0, & x \in (-\infty,-1) \cup (1,\infty) \end{cases} . \quad (39)$$

Proof.

The support is obviously $[-1, 1]$ and thus compact. Smoothness follows from similar arguments to the previous example. □

Remark

By rescaling and and translating the previous example we can obtain a function which is arbitrarily small and with arbitrarily small compact support. Indeed let $\varphi_{\varepsilon,K}$ denote the result of such an operation such that

$$\sup_{x\in K} |\varphi_{\varepsilon,K}(x)| < \varepsilon \qquad \mathrm{supp}(\varphi_{\varepsilon,K}) \subseteq K. \tag{39}$$

We call any such function a *bump function*. Bump functions can be regarded as encoding arbitrarily small smooth but nonanalytic perturbations of ordinary functions. More precisely, for every neighbourhood of a function f in the compact-open topology there is a bump function $\varphi_{\varepsilon,K}$ such that $f + \varphi_{\varepsilon,K}$ also belongs to that neighbourhood.

Theorem

The analytic functions are nowhere dense in any space of functions containing the bump functions and equipped with the compact-open topology.

- Naively, due to the ubiquity of Taylor approximations in the everyday life of a physicist, it might be expected that most physical functions are analytic.
- In fact by the Taylor theorem we only require sufficient differentiability (C^3 tends to be sufficient in practice) and due to the lack of localisable perturbations we may in fact *prefer* to dispense with analyticity assumptions.
- Indeed by the lack of genericity and rigidity of analytic structure it seems appropriate to reserve analyticity assumptions for structures encoded by nondeformable universal structures.

- Complex geometry, especially Calabi-Yau manifolds and twistors.
- KMS condition for thermal states in equilibrium statistical mechanics.
- Analyticity properties of scattering amplitudes follow from universal physical assumptions such as causality and locality.
- Positivity of the energy H in quantum theory implies that the mapping $t \mapsto U_t = \exp(iHt)$ is the boundary value of an analytic (holomorphic) function.
- Wick rotation involves analytic continuation to imaginary time.
- ...

Note:

Analyticity tends to only appear in physics when there are good reasons for it to appear. In the absence of such reasons it is probably safer to assume that analyticity is absent.

Smooth Structures in Physics

There are three primary reasons to use smooth structures in physics:

1. Most physical systems are cast in terms of *differential equations*. Such equations are most naturally formulated for smooth objects.
2. The Taylor expansion applies to smooth functions and gives a quantitative and simple method to consider perturbative corrections to a given system in polynomial form.
3. Rigidity of wider mathematical structures employed:
    - Most continuous group actions are naturally interpreted as smooth since most topological groups are in fact (differentiable) Lie groups (c.f. Hilbert's fifth problem).
    - Local isometries between regions of $\mathbb{R}^D$ equipped with length space metrics associated to quadratic forms are automatically smooth.
    - ..?

The first reason is not very robust:

1. The use of differential equations in physics to express the equations of motion is more a feature of habit and convenience that intrinsic necessity:
   - EoM can often be recast as optimisation problems; such problems have no need to assume smooth structure. For instance the equations of motion are often solved by solutions that could equivalently be characterised via some action/energy minimisation or entropy maximisation property. The equations of motion are *useful* insofar as they allow for simple tests of *single instances* of possible solutions without having to consider the entire feasible set.
   - Differential equations also are often more amenable to study after being cast as integral equations.
   - Similar to the above, solving differential equations may require the employ of weak solutions in less regular function spaces (Besov spaces and Sobolev spaces for instance) than the space of smooth functions. Elementary difficulties with smooth solutions already appear in classical mechanics in connection with e.g. the Lavrentiev phenomenon.

The second reason is more robust if we are interested in approximating via *polynomials*:

Theorem (Converse to the Taylor Theorem)

Let $f : \mathbb{R} \to \mathbb{R}$ be a function such that for each $a \in \mathbb{R}$ there is a polynomial P_a of degree n and a function $\varepsilon_a : \mathbb{R} \to \mathbb{R}$ satisfying

$$\lim_{x \to 0} \varepsilon_a(x)(x-a)^n = 0 \tag{40}$$

such that

$$|f(x) - P_a(x)| = \varepsilon_a(x)(x-a)^n. \tag{41}$$

Then $f \in C^n$.

Remark

Slightly stronger control over the form of the error ε_a allows us to derive a Taylor theorem and converse for functions belonging to the *Hölder spaces* $C^{k,\alpha}$.

Why Polynomials?

Polynomials have several nice mathematical properties:

- Polynomials form an algebra: they are closed under addition, scalar multiplication and multiplication.
- Polynomials are closed under differentiation and integration.
- Polynomials are closed under translation:

$$(x-a)^n = \sum_{\ell+m=n} \frac{n!}{\ell!m!}(b-a)^{\ell}(x-b)^m \tag{42}$$

- Polynomials are simple.

Why Polynomials?

Physically polynomials realise the idea that a *function is homogeneous up to translation*:

- Recall that a function is *homogeneous of degree* α iff

$$f(\lambda x) = \lambda^{\alpha} f(x). \tag{43}$$

- The basic model of such a function is simply the monomial:

$$f(x) = x^{\alpha}. \tag{44}$$

- Lower order monomials may then be generated by translating the centre of expansion.

Fact

Every polynomial has a unique decomposition as a sum of homogeneous polynomials.

Why Polynomials?

The Polynomial Philosophy

- A system described by a polynomial function may be regarded as a superposition (composite) of *homogeneous systems*.
- In particular, polynomials are the smallest closed class of functions that can describe the translates of systems exhibiting power-law scaling (self-similarity) when considered around the appropriate centre.
- Approximately polynomial functions thus describe all approximately self-similar systems.

Definition

A family of functions $\mathscr{F}$ on a set X is said to *separate points* iff for any distinct $x, y \in X$ there is a function $f \in \mathscr{F}$ such that $f(x) \neq f(y)$.

Theorem

Let X be a compact Hausdorff space and let $\mathcal{A}$ be a subalgebra of $\mathcal{C}(X)$ such that $\mathcal{A}$ contains a nonzero constant function. Then $\mathcal{A}$ is dense in $\mathcal{C}(X)$ iff it separates points.

Remark

Since the polynomials contain the constants and separate points this shows polynomials can be used to approximate all continuous functions on a compact space.

- We have a vector space of formal polynomials T. This admits a grading into subspaces T_α containing all polynomials of degree α.
- Every concrete polynomial may then be regarded a *model*

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  - E.g. the abstract polynomial in two variables

$$p(x, y) = x^2 + xy + yx + y^2$$

    can be realised as the either of the polynomials

$$(\Pi_0 p)(a, b) = a^2 + 2ab + b^2 \qquad \Pi_\alpha(a, b) = (a - \alpha)^2 + 2(a - \alpha)(b - \alpha) + (b - \alpha)^2$$

    in $\mathbb{R}$ or as the matrix polynomial

$$(\Pi^{\mathbb{M}} p)(A, B) = A^2 + AB + BA + B^2. \tag{44}$$

- We have a vector space of formal polynomials T. This admits a grading into subspaces T_α containing all polynomials of degree α.
- Every concrete polynomial may then be regarded a *model*

$$\Pi : T \to P(\mathcal{A}). \tag{43}$$

- Let $\mathbf{\Pi}$ denote the space of all models.
- We have translations

$$F_x : T \to T \qquad \Gamma_{xy} = F_x^{-1} \circ F_y \qquad \Pi_x := \Pi \circ F_x \tag{45}$$

  such that if $p \in T_\alpha$, then $F_x p$ is homogeneous near the point x.
- The space of models $\mathbf{\Pi}$ is subject to the *nonlinear* constraints:

$$\Gamma_{xy}\Gamma_{yz} = \Gamma_{xz} \qquad \Pi_y = \Pi_x \circ \Gamma_{xy} \tag{46}$$

- Let us call $(T, \mathbf{\Pi}, \Gamma)$ the *polynomial regularity structure*.
- By the Taylor theorem and its converse, associated to this regularity structure is a canonical class of spaces C^α that we can reconstruct from the space of models **Pi**.

A general regularity structure simply generalises the structural features of the polynomial regularity structure:

Definition

A *regularity structure* in $\mathbb{R}^D$ consists of the following data:

- An *index set* $A \subseteq \mathbb{R}$.
- A graded topological vector space $T = \bigoplus_{a \in A} T_a$.
- A *structure group* G of linear operators $g : T \to T$ such that for each $\alpha \in A$

$$(ga - a) \in \bigoplus_{\beta < \alpha} T_\beta \tag{47}$$

for all $a \in T_\alpha$.

A general regularity structure simply generalises the structural features of the polynomial regularity structure:

Definition

A *model* for a regularity structure (A, T, G) consists of a pair (Γ, Π) where

$$\Gamma : \mathbb{R}^D \times \mathbb{R}^D \to G \qquad \Gamma : (x, y) \mapsto \Gamma_{xy} \tag{48}$$

and $\Pi = \{\, \Pi_x \,\}_{x \in \mathbb{R}^D}$ with

$$\Pi_x : T \to \mathcal{D}^*(\mathbb{R}^D) \tag{49}$$

continuous and linear for each $x \in \mathbb{R}^D$. Moreover we have the following constraints:

$$\Gamma_{xx} = 1_G \qquad \Gamma_{xy}\Gamma_{yz} = \Gamma_{xz} \qquad \Pi_y = \Pi_x \circ \Gamma_{xy} \tag{50}$$

Remark

The key generalisation in the above definitions is that the models Π_a are no longer interpreted as concrete polynomials centred at $a \in \mathbb{R}^D$ but instead can be arbitrary *(tempered) distributions*.

The Definition is Not Really a Definition

The above definition holds 'up to technicalities' which contain the core power of the theory.

Theorem

Let (A, T, G) *be a regularity structure and* (Γ, Π) *a model for* (A, T, G). *Let* $\alpha = \inf A$. *For each* $\gamma \in \mathbb{R}$ *there is a space of modelled distributions* $\mathcal{D}^\gamma$ *consisting of functions*

$$f : \mathbb{R}^D \to \bigoplus_{\alpha<\gamma} T_\alpha \tag{51}$$

such that there is a continuous map

$$\mathcal{R}_\gamma : \mathcal{D}^\gamma \to C^\alpha \tag{52}$$

satisfying the following constraint uniformly on all compact sets: the quantity

$$|(\mathcal{R}_\gamma f - \Pi_x f(x))\phi| \tag{53}$$

is small for all test functions ϕ.

Remark

The *modelled* distributions $\mathcal{D}^\gamma$ generalise the smooth spaces C^α used in traditional analysis and show that the *model* distributions $\Pi_x f(x)$, $t \in T$, can be approximated locally by the 'jets' of distributions f. By appropriately choosing the regularity structure (A, T, G) and the model (Γ, Π) one can thus obtain modelled distributions $\mathcal{D}^\gamma$ tailored to the problem at hand.

- Regularity structures provide a systematic way of generalising the Taylor expansion to study far more singular problems than allowed by the Taylor theorem.
- In particular they allow for the solution to many nonlinear stochastic PDEs because the formalism map automatically provides an algorithm for the multiplication of a wide class of distributions.
- For certain subcritical stochastic PDEs there is a systematic renormalisation procedure for constructing the 'correct' regularity structure+model for the problem under concern.
- Martin Hairer developed this general framework and used it to construct the stochastic quantisation of the ϕ_3^4 theory and solve the KPZ (Kardar-Parisi-Zhang) equation.
- He was awarded the Fields medal in 2014 for this work.

Martin Hairer, *A Theory of Regularity Structures*

We will endow the space of all models … with a topology that enforces the correct behaviour of Π_x near each point x, and furthermore enforces some natural notion of regularity of the map $[\Gamma]$. The important remark is that although this turns the space of models into a complete metric space, it does not turn it into a linear (Banach) space! It is the intrinsic nonlinearity of this space which allows to [sic] encode the subtle cancellations that one needs to be able to keep track of in order to treat the examples mentioned [above].

Part IV

Divergence of Perturbation Theory

Dyson's Argument

- Assume the QED (zero source) partition function admits an analytic series expansion in terms of the fine-structure constant:

$$\mathcal{Z}(\alpha) = \sum_n c_n \alpha^n. \tag{51}$$

- This represents a sum of vacuum diagrams; photon lines are internal so we have an even number of vertices in each diagram: the expansion is in terms of $e^2 = \alpha$.
- Analytically continue to negative α.
- Due to the negative sign of $\alpha = e^2$, like charges Coulomb attract and opposite charges repel.
- The vacuum then admits an instability associated to decay into electron-positron pairs:

- Consider N electrons in a region of radius r and N positrons in a similar region with the two regions separated by a distance $d \gg r$.
- The energy consists of electrostatic repulsion between the two distinct regions, electrostatic attraction internal to each region and kinetic energy coming from linear momentum.
- Since $d \gg r$ we ignore electrostatic repulsion and the electrostatic potential energy is

$$-N^2 \frac{e^2}{r}. \tag{51}$$

- The linear momentum exhibited by the configuration is of order

$$p \sim \frac{N}{r}. \tag{52}$$

- The configuration is then viable as long as

$$N^2 \frac{e^2}{r} > \frac{N}{r} \qquad \text{i.e.} \qquad N \gtrsim \frac{1}{\alpha}. \tag{53}$$

Structure of Dyson's argument:

$$\text{QED at } \alpha < 0 \text{ is unphysical} \implies \text{Perturbation theory diverges.}$$

By contraposition we obtain

$$\text{Perturbation theory converges} \implies \text{QED at } \alpha < 0 \text{ is physical.}$$

Dyson's Assumption on Perturbative Ontology

If the perturbative expansion of a model converges then the solution is physical.

Corollary

The functional relations of physics are analytic.

Example 1 [Herbst and Simon, 1978]

Consider a quantum system with the Hamiltonian:

$$H(g) = p^2 - 1 + \frac{1}{g^2}((gx+1)^2 - 1)^2 - 2gx. \tag{54}$$

The Rayleigh-Schrödinger series $\sum_n a_n g^{2n}$ converges to the incorrect solution.

Proof.

The RS series is defined by $a_n = 0$. On the other hand $E(g) > 0$. To see this first note that:

$$H(g) = A^*(g)A(g) \qquad\qquad A = \frac{\mathrm{d}}{\mathrm{d}x} + x + gx^2$$

so $E(g) \geq 0$.

□

Example 1 [Herbst and Simon, 1978]

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The Rayleigh-Schrödinger series $\sum_n a_n g^{2n}$ converges to the incorrect solution.

Proof.

By a general result $E(g) = 0$ iff the solution f of $Af = 0$ is square-integrable. Taking

$$f(x,g) = \exp\left(-\frac{1}{2}x^2 - \frac{1}{3}x^3 + \frac{1}{6}g^{-2}\right)$$

solves $Af = 0$ and is not L^2. □

Example 2 [Herbst and Simon, 1978]

Consider a quantum system with the Hamiltonian:

$$H(g) = p^2 - 1 + x^2(g^2x^2 + 1)^2 - 3g^2x^x. \tag{55}$$

Though the RS series converges correctly the associated eigenvectors diverge.

Proof.

The RS series is defined by $a_n = 0$. Thus the limit of the eigenvectors is simply the eigenvector of the unperturbed problem (at $g = 0$). This is simply the ground state of a harmonic oscillator, i.e. a Gaussian:

$$f_0(x) = \exp\left(-\frac{1}{2}x^2\right).$$

□

Example 2 [Herbst and Simon, 1978]

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$$H(g) = p^2 - 1 + x^2(g^2x^2 + 1)^2 - 3g^2x^x. \tag{55}$$

Though the RS series converges correctly the associated eigenvectors diverge.

Proof.

To show $E(g) = 0$ again write

$$H(g) = A^*(g)A(g) \qquad A = \frac{\mathrm{d}}{\mathrm{d}x} + x + gx^3$$

so $E(g) \geq 0$ as previously. Use the same general result but now observe that f is L^2. □

Example 2 [Herbst and Simon, 1978]

Consider a quantum system with the Hamiltonian:

$$H(g) = p^2 - 1 + x^2(g^2x^2 + 1)^2 - 3g^2x^x. \tag{55}$$

Though the RS series converges correctly the associated eigenvectors diverge.

Proof.

In particular the function

$$f(x,g) = \exp\left(-\frac{1}{2}x^2 - \frac{1}{4}g^2x^4\right)$$

solves $Af = 0$ and is L^2. f is the ground state of the perturbed system but $f \neq f_0$. □

Example 3 [Simon]

Consider a quantum system with the Hamiltonian:

$$H(g) = p^2 - (1 - g)r^{-1}. \tag{56}$$

For $\lambda \in (\infty, 1)$ the eigenvalues are described by an analytic function that coincides with the Rayleigh-Schrödinger perturbation series. However the Hamiltonian does not admit eigenvalues for $\lambda \geq 1$.

Remark

For $\lambda > 1$ the solutions to perturbation theory can be interpreted as *antibound states*, i.e. poles on unphysical sheets of the scattering amplitude.

Perturbation theory can converge but fail to be physical.

1. This occurs when analyticity assumptions are violated, i.e. the functional relations in question are smooth but not analytic.
   - In this case perturbation theory will converge to an *incorrect answer*.
2. When more than one quantity is treated perturbatively, it is important to check the convergence properties of all quantities in question.
3. Even when analyticity assumptions are not violated, convergence of perturbation theory may be spurious because the domain of analyticity includes unphysical regions.

Jaffe's Modification of Dyson's Argument

- In axiomatic QFT the *spectral condition* says that the Hamiltonian of a valid QFT has a Hamiltonian supported in the forwards light-cone of Minkowski space.
- In $2D$, arguments of Baym allow one to translate vacuum instability (required in Dyson's argument) into violation of the spectral condition.
- Finally observe that perturbation theory satisfies the spectral condition to all orders.
- With this Jaffe argued that perturbation theory *plausibly* diverges. Clearly it either diverges or converges to an unphysical result (that incorrectly respects the spectral condition).
- Jaffe also rigorously proved the divergence of perturbation theory for 2D axiomatic QFTs with the following interaction term:

$$\lambda \sum_{n=3}^{\infty} a_n : \phi^n : \tag{57}$$

Classical Perturbation Theory for Stochastic Quantities Diverges

- Assume we have a physical quantity f depends on some quantity x in a nonlinear manner: $f = f(x)$.
- Moreover suppose that the relation is analytic in some *finite* domain:

$$f(x) = \sum_n a_n x^n \tag{58}$$

for all $x \in \mathbb{D}$.

- Finally assume that x is a stochastic quantity described by e.g. a Gaussian random variable with standard deviation σ.

Classical Perturbation Theory for Stochastic Quantities Diverges

- By the final assumption

$$\langle x^n \rangle = \begin{cases} 0, & n \text{ is odd} \\ \sigma^{2k}(2k-1)!! & n = 2k \end{cases} \tag{58}$$

where $(2k-1)!! = (2k-1)(2k-3)\cdots(3)(1)$.

- It can be shown that

$$(2k-1)!! = \frac{(2k-1)!}{2^{k-1}(k-1)!} \tag{59}$$

i.e.

$$(2k-1)!! \sim \exp\left(2k\ln 2k - k\ln 2 - k\ln k\right) = \exp(k\ln 2k) \tag{60}$$

for large k by Stirling's theorem.

- Thus

$$\langle f(x)\rangle_{PT} := \sum_n a_n \langle x^n \rangle = \sum_k a_{2k}\sigma^{2k}(2k-1)!! = \sum_k a_{2k}\sigma^{2k}\exp(k\ln 2k) \tag{61}$$

and the perturbative expansion of $\langle f(x)\rangle$ diverges.

- f is an analytic function of the random variable x and so $\langle f(x) \rangle$ is well-defined.
- In particular

$$\langle f(x) \rangle \neq \langle f(x) \rangle_{PT} \tag{62}$$

since the LHS exists and the RHS diverges.
- What is the problem?

- Naively we might hope that:

$$\left\langle \sum_n a_n x^n \right\rangle \stackrel{?}{\neq} \sum_n a_n \langle x^n \rangle . \tag{62}$$

- In fact the above is fine up to technicalities (continuity assumptions etc).

- More problematic is

$$\langle f(x)\rangle \stackrel{?}{=} \left\langle \sum_n a_n x^n \right\rangle . \tag{62}$$

- For the above to hold we need the function f to be analytic in the *entire* support of the measure describing the expectation $\langle\cdot\rangle$.
- In general the function f will only be analytic in a small region $\mathbb{D}$ of the support of the law of f and so we cannot compute the expectation via the analytic series expansion.

- We have thus accounted for the inequality $\langle f \rangle \neq \langle f \rangle_{PT}$.
- We have not, however, explained the *divergence* of $\langle f \rangle_{PT}$.
- This comes from the structure of the moments in for Gaussian random variables.
- This is *not* a problem special to Gaussian random variables.
- The challenge is to ensure that

$$\langle f \rangle_{PT} = \sum_n a_n \langle x^n \rangle \tag{62}$$

  by controlling the growth of the moments $\langle x^n \rangle$.
- If the radius of convergence of f is ≤ 1 then the moments are at least *uniformly bounded*:

$$\langle x^n \rangle \leq K \qquad \forall n. \tag{63}$$

- This only happens if the law of f has support in $[-1, 1]$.

Summary

- Classical perturbation theory diverges due to chaos (sensitivity to initial conditions) and resonances.
- Dyson's argument from quantum field theory explains why perturbation theory *plausibly* diverges.
- Because perturbation theory can converge to incorrect/unphysical solutions Dyson's argument is not the last word on the topic.
- On very general grounds we expect perturbation theory to fail to converge to the correct solution for most nonlinear stochastic relations with *limited* analyticity properties.

Part V

Asymptotic Expansions

Overview I

- Perturbation theory gives *asymptotic expansions* to underlying smooth (analytic) objects.
- There are several senses in which a series is asymptotic to some underlying function: in particular we identify as distinct the notions of *asymptotic*, *strongly asymptotic* and *Gevrey-s asymptotic*.
- While an ordinary asymptotic expansion is a power series that is uniquely determined for every smooth real function, the converse does not hold: the same series is asymptotic to multiple functions.
- On the other hand, if a power series is *strongly asymptotic* to analytic some smoooth function then that smooth function is unique.
- *Gevrey-s functions* interpolate between analytic functions (for $s = 1$) and general smooth functions (at large s).
- *Gevrey-s series* are a special type of formal series satisfying a growth condition on the coefficients.

Overview II

- If a function is *Gevrey-s asymptotic* to a series, then that series obeys the Gevrey-s growth condition on its coefficients.
- Functions that are Gevrey asymptotic to Gevrey series obey the remarkable property that there is some finite truncation of the asymptotic Gevrey series that captures the function up to an *exponentially small* error.
- The *Neishtadt theorem* states that the formal solution to a perturbation of a classical integrable system exists and solves the perturbed system up to an *exponentially small error*.

Question: Why does perturbation theory work?
Answer: because perturbation theory defines an asymptotic expansion of an analytic function.

Definition

Let $f : \mathbb{C} \to \mathbb{C}$ be a function. A series

$$\Sigma_f := \sum_n a_n (z-a)^n \tag{64}$$

is said to be an *asymptotic expansion* of f at a iff for each $N \in \mathbb{N}$

$$\lim_{z \to a} (z-a)^{-N} \left(f(z) - \sum_{n=0}^{N} a_n (z-a)^n \right) = 0. \tag{65}$$

Asymptotic Expansions

Problem

Prove that a smooth function has at most one asymptotic expansion.

Problem

Prove that a smooth function has at most one asymptotic expansion.

Rearranging equation 71, we see that

$$a_N = \lim_{z \to a} (z-a)^{-N} \left(f(z) - \sum_{n=0}^{N-1} a_n (z-a)^n \right) \tag{66}$$

and the series can be specified from f recursively. □

Problem

Prove that the Taylor expansion of every real smooth function is an asymptotic expansion of the function.

Hint: use the uniform estimate for the remainder.

Problem

Prove that the Taylor expansion of every real smooth function is an asymptotic expansion of the function.
Hint: use the uniform estimate for the remainder.

Noting that by the Hadamard lemma $f(x) - P_n(x) = x^n h_n(x)/n!$ where P_n is the Taylor polynomial we find that

$$\lim_{x \to 0} x^{-n}(f - P_n(x)) = \lim_{x \to 0} \left(\frac{x^{n+1} M}{x^n (n+1)!} \right) = 0. \tag{67}$$

□

Problem

Let f and g be functions with the asymptotic expansions $\Sigma_f = \sum_n a_n z^n$ and $\Sigma_g = \sum_n b_n z^n$ respectively.

1. Show that the functions $f \pm g$ have the asymptotic expansions $\Sigma_{f \pm g} = \sum_n (a_n \pm b_n) z^n$ respectively.
2. Show that the product fg has an asymptotic expansion $\Sigma_{fg} = \sum_n c_n z^n$ where

$$c_n = \sum_{k=0}^{n} a_k b_{n-k}. \tag{68}$$

1. Simply note that

$$\lim_{z\to 0} z^{-N}\left((f\pm g)(z)-\sum_{n=0}^{N}(a_n\pm b_n)z^n\right)=\lim_{z\to 0} z^{-N}\left(f-\sum_{n=0}^{N}a_n z^n\right)+\lim_{z\to 0} z^{-N}\left(g-\sum_{n=0}^{N}b_n z^n\right)=0.$$

□

2. Since

$$\sum_{n=0}^{N} c_n z^n = \sum_{n=0}^{N} \sum_{k=0}^{n} a_n b_{n-k} z^n = \sum_{k=0}^{N} a_k z^k \sum_{n=k}^{N} b_{n-k} z^{n-k}$$

we have

$$\begin{aligned} fg - \Sigma_{fg} &= fg - \sum_{k=0}^{N} a_k z^k \sum_{n=k}^{N} b_{n-k} z^{n-k} \\ &= g\left(f - \sum_{k=0} a_k z^k\right) + \sum_{k=0}^{N} a_k z^k \left(g - \sum_{n=0}^{N-k} b_n z^n\right). \end{aligned}$$

Substituting into the asymptotic condition and applying continuity proves the statement. □

Problem

Let f be a smooth function with asymptotic expansion $\sum_n a_n x^n$. Show that f' has an asymptotic expansion $\sum_n (n+1) a_{n+1} x^n$.

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Let f be a smooth function with asymptotic expansion $\sum_n a_n x^n$. Show that f' has an asymptotic expansion $\sum_n (n+1)a_{n+1}x^n$.

Since f is smooth, f' is also smooth and thus its Taylor series is its asymptotic expansion. Computing terms and comparing given that $a_n = f^n(0)/n!$ proves the statement. □

Problem

1. Assume that $\Sigma := \sum_n a_n x^n$ is an asymptotic expansion of some function $f(x)$. Show that Σ is an asymptotic expansion of

$$g(x) = f(x) + \alpha \exp\left(-\frac{1}{\alpha x^2}\right) \tag{68}$$

for all values $\alpha > 0$.

2. Find a value of α such that $|f(x) - g(x)| > 10^4$ for all $x \in (0.1, 100)$.

1. Simply recall that $\exp(-1/x^2)$ has 0 as its asymptotic expansion. 2. Choosing α such that $x\alpha \gg 1$ in the specified region of x ensures the exponential is approximately 1. □

Thus we have seen that while every smooth function has a unique asymptotic expansion, the same series can be asymptotic for many rather different functions. To see when a series characterises a function directly we use the following theorem:

Theorem

Carleman Let $f : \mathbb{C} \to \mathbb{C}$ be a function analytic in the region

$$\mathbb{D} := \left\{ z \in \mathbb{C} : |z| < r \text{ and } |\arg(z)| < \frac{\pi}{2} \right\} \tag{68}$$

and continuous on the closure of $\mathbb{D}$. Suppose there is a $K \in (0, \infty)$ such that

$$|f(z)| \leq K^{n+1} n! |z|^n \tag{69}$$

for all n. Then $f = 0$.

Problem

Prove that an entire function satisfying the estimate in Carleman's theorem on some open neighbourhood of zero is identically zero.

Problem

Prove that an entire function satisfying the estimate in Carleman's theorem on some open neighbourhood of zero is identically zero.

Since f is analytic take $f(z) = \sum_k a_k z^k$. We show that $a_k = 0$ and apply the fundamental lemma. In particular by the inequality 69 we have that $\lim_{z\to 0} |f(z)| \leq 0$ i.e. $f(0) = 0$ by continuity. Hence $a_0 = 0$. Assume that $a_0, ..., a_\ell = 0$ so that

$$f(z) = \sum_{k=\ell+1}^{\infty} a_k z^k = z^{\ell+1} \sum_{k=0}^{\infty} a_{k+\ell+1} z^k.$$

Problem

Prove that an entire function satisfying the estimate in Carleman's theorem on some open neighbourhood of zero is identically zero.

By inequality 69 for $n > \ell + 1$ we thus have

$$\left| \sum_{k=0}^{\infty} a_{k+\ell+1} z^k \right| \leq K^{n+1} n! |z|^{n-\ell-1}$$

so by taking $z \to 0$ we see that $|a_{\ell+1}| \leq 0$ i.e. $a_{\ell+1} = 0$ as required. $\square$

Definition

Let f be a complex function. A series

$$\Sigma_f := \sum_n a_n z^n \tag{70}$$

is said to be *strongly asymptotic* to f iff f is analytic in a region of the form $\mathbb{D}$ above and there is an $A \in (0, \infty)$ such that for each $N \in \mathbb{N}$

$$\left| f(z) - \sum_{n=0}^{N} a_n z^n \right| \leq A^{N+1} N! |z|^{N+1} \tag{71}$$

for all $z \in \mathbb{D}$.

Problem

Show that a strongly asymptotic expansion of f is an asymptotic expansion of f.

Problem

Show that a strongly asymptotic expansion of f is an asymptotic expansion of f.

Note that the RHS of the strong asymptotic expansion inequality depends on $|z|^{N+1}$ so dividing by $|z|^N$ on both sides and taking the limit $z \to 0$ gives 0 on the RHS.

Strong Asymptotic Expansions

Problem

Let $\Sigma_1 = \sum_n a_n z^n$ and $\Sigma_2 = \sum_n b_n z^n$ be strongly asymptotic to f. Then $\Sigma_1 = \Sigma_2$.

Problem

Let $\Sigma_1 = \sum_n a_n z^n$ and $\Sigma_2 = \sum_n b_n z^n$ be strongly asymptotic to f. Then $\Sigma_1 = \Sigma_2$.

Note that

$$\left|\sum_{n=0}^{N}(a_n - b_n)z^n\right| \leq \left|f(z) - \sum_{n=0}^{N} a_n z^n\right| + \left|f(z) - \sum_{n=0}^{N} b_n z^n\right| \leq A^{N+1} N! |z|^{N+1}$$

so $\sum_{n=0}^{N}(a_n - b_n)z^n$ is identically zero by Carleman's theorem, i.e. $a_n = b_n$ for all $n \leq N$ for all N. But this implies that $\Sigma_1 = \Sigma_2$.

Problem

Let f and g have the same strongly asymptotic expansion $\Sigma = \sum_n a_n z^n$. Show that $f = g$.

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Let f and g have the same strongly asymptotic expansion $\Sigma = \sum_n a_n z^n$. Show that $f = g$.

Note that

$$|f(z) - g(z)| = |f(z) - \Sigma + \Sigma - g(z)| \leq |f(z) - \Sigma| + |g(z) - \Sigma|$$

and so $f - g$ satisfies the assumptions of the Carleman theorem, i.e. it is identically zero.

Fact [Cauchy Integral Formula]

Let $f = f(z)$ be a complex function analytic in some open subset of $\mathbb{C}$ containing the closed disk of radius r centred at a:

$$\overline{\mathbb{D}}_{r,a} := \{ z \in \mathbb{C} : |z - a| \leq r \} . \tag{72}$$

Let γ denote the closed curve bounding $\mathbb{D}$ with an anticlockwise orientation. Then for every $z \in \mathbb{D}_{r,a} := \overline{\mathbb{D}}_{r,a} \backslash \gamma$ we have

$$f(z) = \frac{1}{2\pi i} \oint_{\gamma} \mathrm{d}\tilde{z} \frac{f(\tilde{z})}{\tilde{z} - z} . \tag{73}$$

Problem [Cauchy Estimate]

Using the Cauchy integral formula prove that every complex analytic function f defined in the disk $\mathbb{D}_{r,a}$ as above satisfies

$$|f^n(z)| \leq \frac{n!}{r^n} \sup_{\tilde{z}\in\mathbb{D}_{r,a}} |f(\tilde{z})|. \tag{74}$$

Problem [Cauchy Estimate]

Using the Cauchy integral formula prove that every complex analytic function f defined in the disk $\mathbb{D}_{r,a}$ as above satisfies

$$|f^n(z)| \leq \frac{n!}{r^n} \sup_{\tilde{z} \in \mathbb{D}_{r,a}} |f(\tilde{z})|. \tag{74}$$

Differentiate under the integral sign in the Cauchy formula and substitute the supremum for f in the integrand. Evaluate the remaining integral.

Problem

1. Let f and g have the strong asymptotic expansions Σ_f and Σ_g. Show that $f \pm g$ and fg have the natural strong asymptotic expansions.
2. Let f have a strong asymptotic expansion $\sum_n a_n z^n$. Prove that $f'(z)$ has the strong asymptotic expansion $\sum_n (n+1) a_{n+1} z^n$.
   Hint: Use the Cauchy estimate.

Strong Asymptotic Expansions

1. The first statement is a trivial consequence of the subadditivity of the norm. For the second statement use the expression for $fg - \Sigma_{fg}$ deduced previously and apply the triangle inequality to the norm:

$$\begin{aligned} |fg - \Sigma_{fg}| &= \left| g\left(f - \sum_{k=0}^{N} a_k z^k\right) + \sum_{k=0}^{N} a_k z^k \left(g - \sum_{n=0}^{N-k} b_n z^n\right)\right| \\ &\leq |g| \cdot \left| f - \sum_{k=0}^{N} a_k z^k \right| + \sum_{k=0}^{N} |a_k| \cdot |z|^k \cdot \left| g - \sum_{n=0}^{N-k} b_n z^n \right| \\ &\leq |g| \cdot A^{N+1} N! |z|^{N+1} + \sum_{k=0}^{N} |a_k| A_k^{N-k+1} (N-k)! |z|^{N+1}. \end{aligned}$$

Noting that $|g|$ is bounded in the region $\mathbb{D}$ we can then choose constants ensuring the desired result. □

2 Define $g = f - \sum_{n=0}^{N} a_n z^n$. Using the Cauchy estimate for g' for a circel $\mathbb{D}_{r,a} \subseteq \mathbb{D}$ thus gives

$$\left| f' - \sum_n (n+1) a_{n+1} z^n \right| \leq \frac{1}{r} \sup_{\tilde{z} \in \mathbb{D}_{r,a}} |g(\tilde{z})| \leq \frac{1}{r} \sup_{\tilde{z} \in \mathbb{D}} |g(\tilde{z})| \leq \frac{1}{r} A^{N+1} N! |z|^{N+1}$$

as required.

Definition

Let $U \subseteq \mathbb{R}$ be open. A mapping $f : U \to \mathbb{R}$ is said to be *Gevrey-σ* iff for each compact set $K \subseteq U$ we have constants $M_K, N_K \in (0, \infty)$ such that

$$\sup_{x \in K} |f^n(x)| \leq M_K C_k^n (n!)^\sigma. \tag{75}$$

Remark

Immediately note that if f is Gevrey-s then it is Gevrey-t for all $t > s$.

Problem [Analytic Implies Gevrey-1]

Using the Cauchy estimate prove that a real function analytic on an interval is Gevrey-1.

Hint: every real function analytic on an interval admits for each compact K strictly contained in the interval a point $x \in K$ and a unique analytic continuation to an open disk in $\mathbb{C}$ centred at x containing K.

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By the hint for each compact K in the analytic interval of f we have an analytic continuation to some disk centred about a point $x \in K$ such that

$$|f^{(n)}(z)| \leq M_K \left(\frac{1}{r}\right)^n n! \tag{76}$$

where $M_K = \sup_{z \in \mathbb{D}_{r,x}} |f(z)|$. Taking $C_K = (1/r)$ gives the desired result. □

Problem [Gevrey-1 Implies Analytic]

Show that every Gevrey-1 function is analytic.
Hint: use the Lagrange form of the remainder in the Taylor theorem.

Problem

Show that the class of Gevrey-s functions is closed under addition and multiplication.
Hint: Use the Leibniz formula

$$(fg)^{(n)} = \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} f^{(k)} g^{(n-k)}. \tag{77}$$

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Addition is trivial. For multiplication simply note that

$$\begin{aligned} \left|(fg)^{(n)}\right| &= \left|\sum_{k=0}^{n} \frac{n!}{k!(n-k)!} f^{(k)} g^{(n-k)}\right| \le \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} |f^{(k)}| \cdot |g^{(n-k)}| \\ &\le \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} M_K^f C_{f,K}^k (k!)^s M_K^g C_{g,K}^{n-k} ((n-k)!)^s \le \left(\sum_{k=0}^{n} \frac{n!}{k!(n-k)!}\right) C^k (n!)^s \end{aligned} \tag{78}$$

where $C > C_{f,K},\ C_{g,K}$. □

Problem

Show that the derivative of a Gevrey-s function is Gevrey-s.

Fact [Cauchy-Hadamard]

The formal power series

$$f(z) = \sum_n a_n (z-a)^n \tag{79}$$

has a radius of convergence R where

$$\frac{1}{R} = \limsup_{n\to\infty} \left(|a_n|^{\frac{1}{n}} \right) := \lim_{n\to\infty} \sup_{m\geq n} \left(|a_m|^{\frac{1}{m}} \right) \tag{80}$$

Problem

Using the Cauchy-Hadamard theorem show that every real Gevrey-s function with $s \in (0,1)$ is entire i.e. analytic everywhere.

Hint: Use the fact that

$$n! \geq z\left(\frac{n}{e}\right)^{n}. \tag{81}$$

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Since every Gevrey-s function is Gevrey-1 for $s < 1$ we can fix a compact set K such that

$$f(z) = \sum_n \frac{1}{n!} f^{(n)}(z_0)(z - z_0)^n$$

for $z_0 \in K$ and

$$|a_n|^{\frac{1}{n}} = \left|\frac{1}{n!} f^{(n)}(z_0)\right|^{\frac{1}{n}} \leq \left(M_K C_K^n (n!)^{s-1}\right)^{\frac{1}{n}} = C_K M_K^{\frac{1}{n}} (n!)^{\frac{s-1}{n}}.$$

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Since $n! \geq z\left(\frac{n}{e}\right)^n$, $(n!)^{1/n} \geq n/e$ and $(n!)^{1/n} \to \infty$ as $n \to \infty$. Hence $(n!)^{\frac{s-1}{n}} \to 0$ as $n \to \infty$ and

$$\limsup_{n\to\infty}\left(|a_n|^{\frac{1}{n}}\right) = 0$$

i.e. $R = \infty$ as required. □

Definition

A series $\Sigma = \sum_n a_n x^n$ will be called *Gevrey-s* iff there are positive constants K, A, $\alpha \in (0, \infty)$ such that

$$|a_n| \leq KA^{sn}\Gamma(sn + \alpha). \tag{82}$$

Remark

Recall that $\Gamma(n+1) = n!$ indicating that the above is simply a generalisation of the standard Gevrey inequality.

Definition

An *open sector* of $\mathbb{C}$ is any set

$$\{ z \in \mathbb{C} : \alpha < \arg(z) < \beta \} \tag{83}$$

for suitable numbers α and β. A *closed sector* is similarly defined except the inequalities are not strict:

$$\{ z \in \mathbb{C} : \alpha \leq \arg(z) \leq \beta \} . \tag{84}$$

Note that we assume 0 belongs to each sector by assumption.

Definition

Let $f(z)$ be a complex function and let $\Sigma := \sum_n a_n z^n$ a formal power series. Let V be a sector. f is *Gevrey-s asymptotic to* Σ iff there is a positive constant α, $A > 0$ and for each closed subsector W of V there is a positive constant B_W such that

$$\left| f(z) - \sum_{n=0}^{N-1} a_n z^n \right| \leq B_W A^{sN} \Gamma(sN + \alpha) |z|^N \tag{85}$$

for all integers $N > 0$ for all $z \in W$. A function analytic in V that is Gevrey-s asymptotic to some formal power series will be called $\mathcal{A}_s(V)$.

Problem

Show that if f is $\mathcal{A}_s(V)$ for some open sector V then any formal series Σ that is Gevrey-s asymptotic to f is a Gevrey-s series.

Note: we now take the sets K given in the definition of Gevrey functions to be closed sectors.

Definition

Let $s = 1/k > 0$ and V be an open sector. A function f has $\sum_n a_n z^n$ as a *cutoff asymptotic* of order s iff there are positive constants A, $\alpha > 0$ and for each closed sector $W \subset V$ a positive constant $C_W \in (0, \infty)$ such that

$$\left| f(x) - \sum_{0 \leq n \leq kA^{-1}|z|^{-k}} a_n z^n \right| \leq C_W |z|^\alpha \exp\left(-\frac{1}{A|z|^k} \right) \qquad (86)$$

for all $z \in W$.

Problem

Show that if f is an analytic function admitting a cutoff asymptotic of order s then its asymptotic expansion is a Gevrey-s series.

Theorem (Ramis and Schäfke)

A function f analytic on an open sector V has a Gevrey-s asymptotic expansion iff it has a cutoff asymptotic of order f on V.

Part VI

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