# Resummation of divergent series


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Asymptotics in astrophysics
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# This talk

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Techniques to resum non-convergent series

*ubiquitous!*

- Borel resummation
- Resurgence
- Lefschetz thimble

∃ Many possible applications in various contexts

# Different expansions have different stories…

Physical setup:

Field Theory, Gravity, Statistical system, etc… ?

Expansion parameters:

Coupling constant, $N$, $N_f$, $\alpha'$, time, $T$, $\mu$, $\epsilon$ ,etc… ?

around…?

$0$, $\infty$, or finite point… ?

Technical setup:

(path) integral or differential/difference eq… ?

# Contents

# Non-convergent series

Formal power series: $$\sum_{n=0}^{\infty} c_n z^n$$

Radius of convergence: $$r_c = \lim_{n \to \infty} \left| \frac{c_n}{c_{n+1}} \right|$$

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Ex.1) $e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$

$$r_c = \lim_{n\to\infty} \left| \frac{(n+1)!}{n!} \right| = \infty$$

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Ex.2) $\frac{1}{1-az} = \sum_{n=0}^{\infty} (az)^n$ $\qquad$ $r_c = \lim_{n\to\infty} \left|\frac{a^n}{a^{n+1}}\right| = \frac{1}{|a|}$

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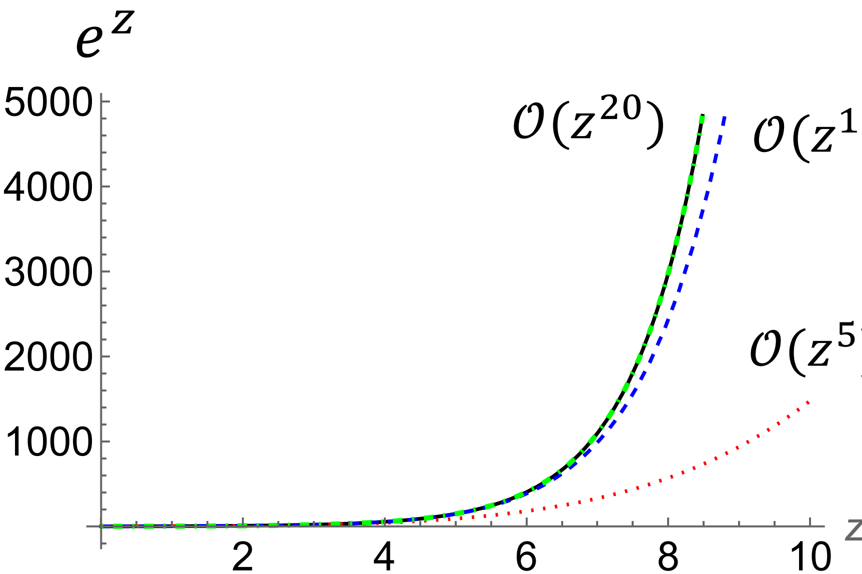
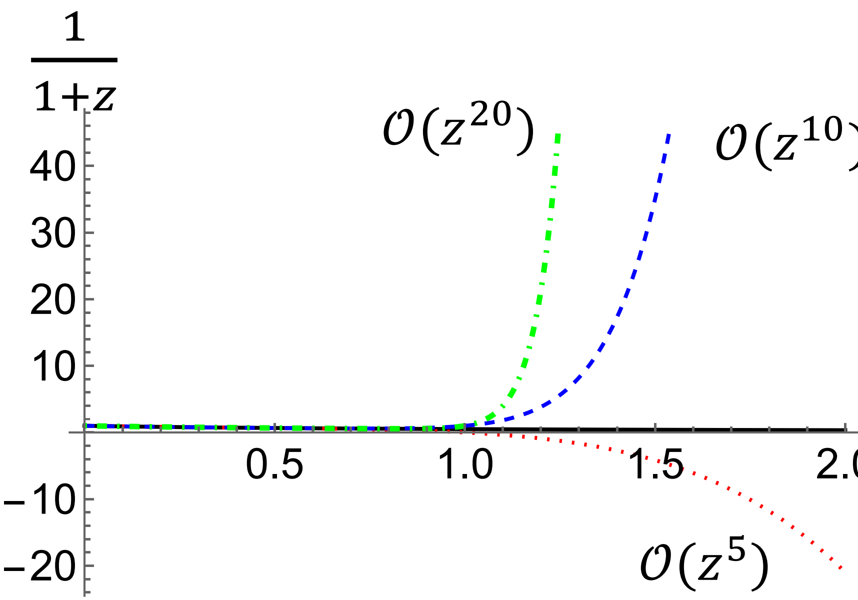
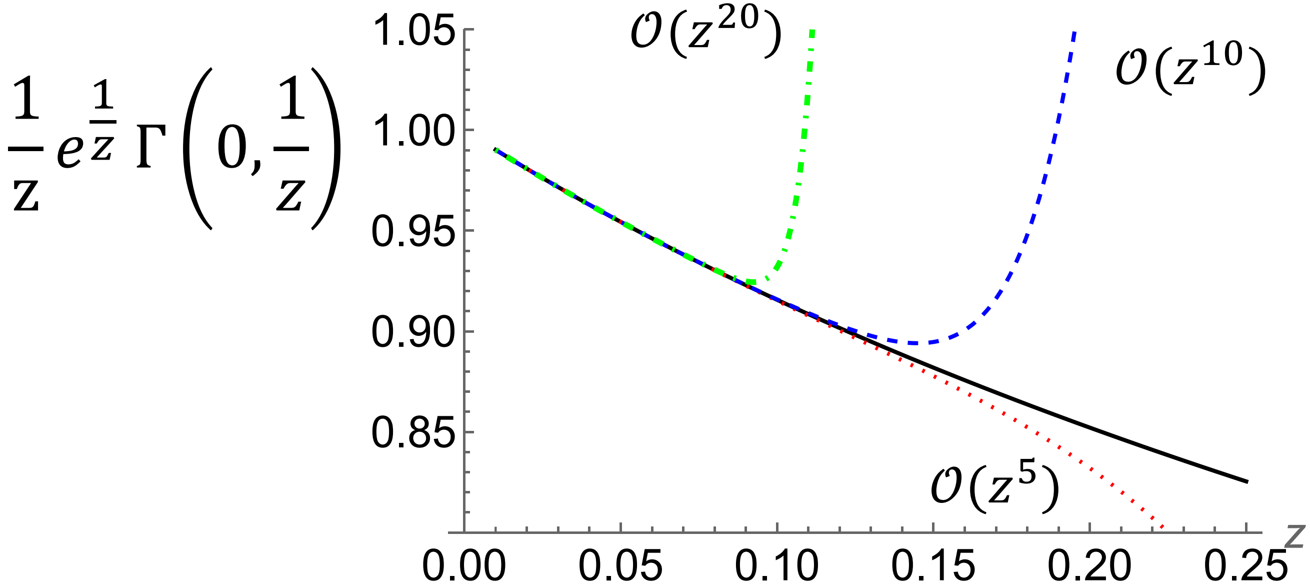
Ex.3) $\frac{1}{z} e^{\frac{1}{z}} \Gamma\left(0, \frac{1}{z}\right) \simeq \sum_{n=0}^{\infty} n!\,(-z)^n$ $\qquad r_c = \lim_{n\to\infty} \left|\frac{(-1)^n n!}{(-1)^{n+1}(n+1)!}\right| = 0$

*non-convergent!*

# Non-convergent series (cont'd)



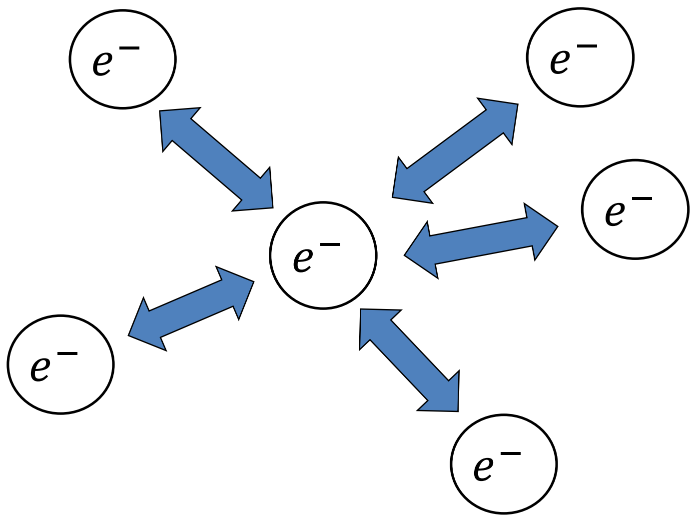



# Perturbative series in QFT is not convergent

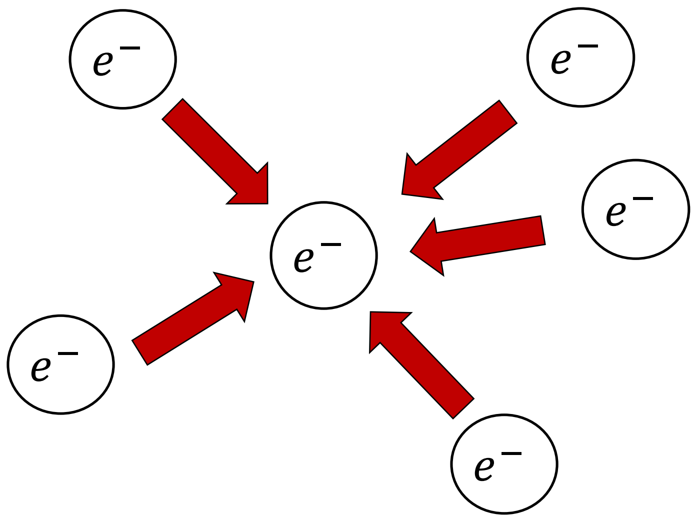
～Dyson's original argument (very rough)～ [Dyson '52]

World w/ $e^2 > 0$




*repulsive*

World w/ $e^2 < 0$




*attractive, prefer to be dense*

looks qualitatively different ⟹ non-analytic?

# Perturbative series in QFT is not convergent

～technical reasons～

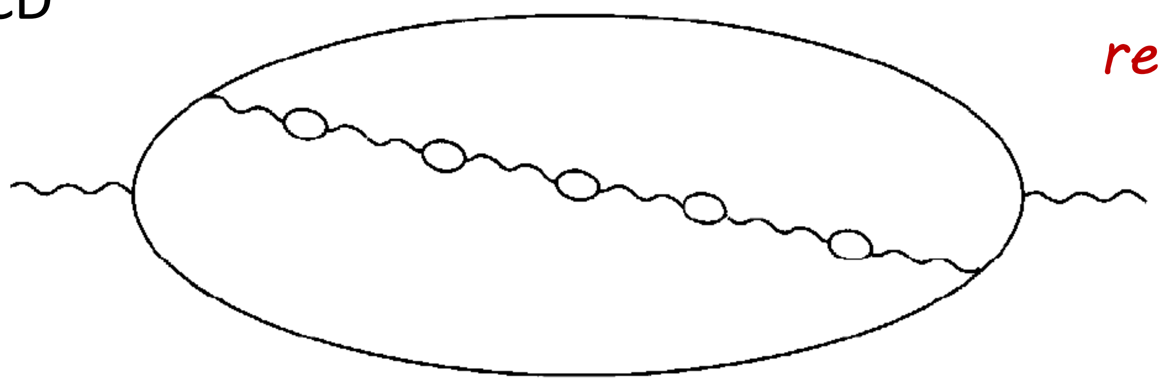
① (# of n-loop Feynmann diagrams) ～ n!

*proliferation*

② ∃Feynmann diagrams contributing by ～n!

Ex.) QCD

*renormalon*



[ Fig.20.3 in Weinberg's book,
cf. Takaura-san's lectures ]

# Best way by Naïve sum = Truncation

$N$-th order approximation of a function $P(g)$:

$$P_N(g) \equiv \sum_{\ell=0}^{N} c_\ell \, g^\ell$$

"error" of the approximation:

$$\delta_N(g) \equiv P_{N+1}(g) - P_N(g) = c_{N+1} g^{N+1}$$

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Optimized order $N_*$: (given $g$)

$$\frac{\partial}{\partial N} \delta_N(g) \Big|_{N=N_*} = 0 \quad \xrightarrow{N \gg 1} \quad \frac{\partial}{\partial N} (\log c_N + N \log g) \Big|_{N=N_*} = 0$$

# Best way by Naïve sum = Truncation (Cont'd)

$$P_N(g) \equiv \sum_{\ell=0}^{N} c_\ell \, g^\ell \quad \xrightarrow{\text{optimize}} \quad \frac{\partial}{\partial N}(\log c_N + N \log g)_N \Big|_{N=N_*} = 0$$

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In QFT, typically

$$c_\ell \sim \ell! \, A^\ell \quad (\ell \gg 1)$$

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In QFT, typically

$$c_\ell \sim \ell!\, A^\ell \quad (\ell \gg 1)$$

Then,

$$0 = \frac{\partial}{\partial N}\left(N\log N - N + N\log(\mathrm{Ag})\right)\Big|_{N=N_*} \quad \Rightarrow \quad N_* = \frac{1}{Ag}$$

Error of the truncation:

$$\delta_{N_*}(g) = c_{N_*+1} g^{N_*+1} \sim e^{-N_*} = \underline{e^{-\frac{1}{Ag}}}$$

Non-perturbative effect

Is there a good way to resum perturbative series?

# General questions

- What does perturbative series actually know?
- Is there a way to obtain exact answer from information on perturbative expansion?
- If yes, how?

# More precise (but still imprecise) question

Perturbative series around saddle points:

$$\mathcal{O}(g) \simeq \sum_{\ell=0}^{\infty} c_\ell^{(0)} g^\ell + \sum_{I \in \text{saddles}} e^{-S_I(g)} \sum_{\ell=0}^{\infty} c_\ell^{(I)} g^\ell$$

Can we get the exact result by using the coefficients?

= What is a correct way to resum the perturbative series?

(∼definition of quantum field theory?)

This talk = to give a partial answer

# Contents

# A standard resummation

## Borel transformation:

$$\mathcal{O}(g) \simeq \sum_{\ell=0}^{\infty} c_\ell g^{a+\ell} \quad \Rightarrow \quad \boxed{\mathcal{B}\mathcal{O}(t) = \sum_{\ell=0}^{\infty} \frac{c_\ell}{\Gamma(a+\ell)} t^{a+\ell-1}}$$

## Borel resummation (along θ):

$$\boxed{S_\theta \mathcal{O}(g) = \int_0^{e^{i\theta}\infty} dt\ e^{-\frac{t}{g}}\ \mathcal{B}\mathcal{O}(t)}$$ (usually, $\theta = \arg(g) = 0$)

# Why Borel resummation may be nice

( Let's take $\theta = \arg(g)$ )

$$S_\theta \mathcal{O}(g) = \int_0^{e^{i\theta}\infty} dt\ e^{-\frac{t}{g}}\ \mathcal{B}\mathcal{O}(t) \qquad \mathcal{B}\mathcal{O}(t) = \sum_{\ell=0}^{\infty} \frac{c_\ell}{\Gamma(a+\ell)} t^{a+\ell-1}$$

① Reproduce original perturbative series:

$$S_\theta \mathcal{O}(g) \simeq \sum_{\ell=0}^{\infty} \frac{c_\ell}{\Gamma(a+\ell)} \int_0^{e^{i\theta}\infty} dt\ t^{a+\ell-1} e^{-\frac{t}{g}} = \sum_{\ell=0}^{\infty} c_\ell g^{a+\ell}$$

②

# Why Borel resummation may be nice

( Let's take $\theta = \arg(g)$ )

$$S_\theta \mathcal{O}(g) = \int_0^{e^{i\theta}\infty} dt\ e^{-\frac{t}{g}}\ \mathcal{BO}(t) \qquad \mathcal{BO}(t) = \sum_{\ell=0}^{\infty} \frac{c_\ell}{\Gamma(a+\ell)} t^{a+\ell-1}$$

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② Finite for any $g$ if

1. Borel trans. is convergent
2. Its analytic continuation does not have singularities along the contour
3. The integration is finite

"Borel summable (along $\theta$)"

related to exact result?

# Some simple examples

1. Analytic function

$$\mathcal{O}(g) = \sum_{\ell} c_\ell g^\ell$$ convergent inside radius of convergence

= (Borel resummation)

2.

# Some simple examples

## 1. Analytic function

$$\mathcal{O}(g) = \sum_{\ell} c_\ell g^\ell$$ convergent inside radius of convergence

= (Borel resummation)

## 2. Incomplete gamma function

$$\mathcal{O}(g) = \frac{1}{g} e^{\frac{1}{g}} \Gamma\left(0, \frac{1}{g}\right) \sim \sum_{\ell} \ell! (-g)^\ell$$

$$\Rightarrow \quad \mathcal{B}\mathcal{O}(t) = \sum_{\ell=0} (-t)^\ell = \frac{1}{1+t}$$ Borel summable along $\boldsymbol{R}_+$

$$S_0 \mathcal{O}(g) = \frac{1}{g} \int_0^\infty dt\, e^{-\frac{t}{g}} \mathcal{B}\mathcal{O}(t) = \frac{1}{g} \int_0^\infty dt\, \frac{e^{-\frac{t}{g}}}{1+t} = \mathcal{O}(g)$$

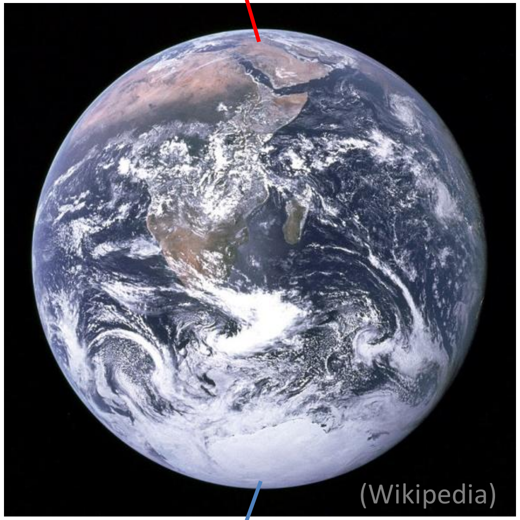
# Physical examples: 4d extended SUSY theories

**Set up:** (similar for 5d N=1 case)

- Theories w/ $\beta \leqq 0$ and Lagrangians $(Z_{S^4} < \infty)$
- Perturbative expansion by $g_{YM}$ around fixed # of instanton/anti-inst.

[MH '16]

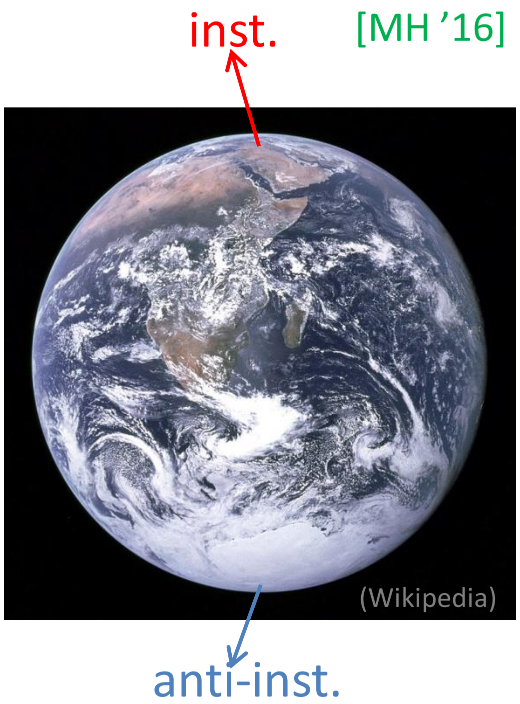
inst.




anti-inst.

**Results:**

# Physical examples: 4d extended SUSY theories

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**Set up:** (similar for 5d N=1 case)

- Theories w/ β≦0 and Lagrangians $(Z_{S^4} < \infty)$
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**Results:** [cf. some low rank cases: Russo, Aniceto-Russo-Schiappa, Gerchkovitz-Gomis-Ishtiaque-Karashik-Komargodski-Pufu ]

- Find explicit finite dimensional integral rep. of Borel trans. for various observables
- ∃ Singularities only along R- → Borel summable along R+
- (Exact) = $\sum_{\text{instantons}}$ (Borel resum)

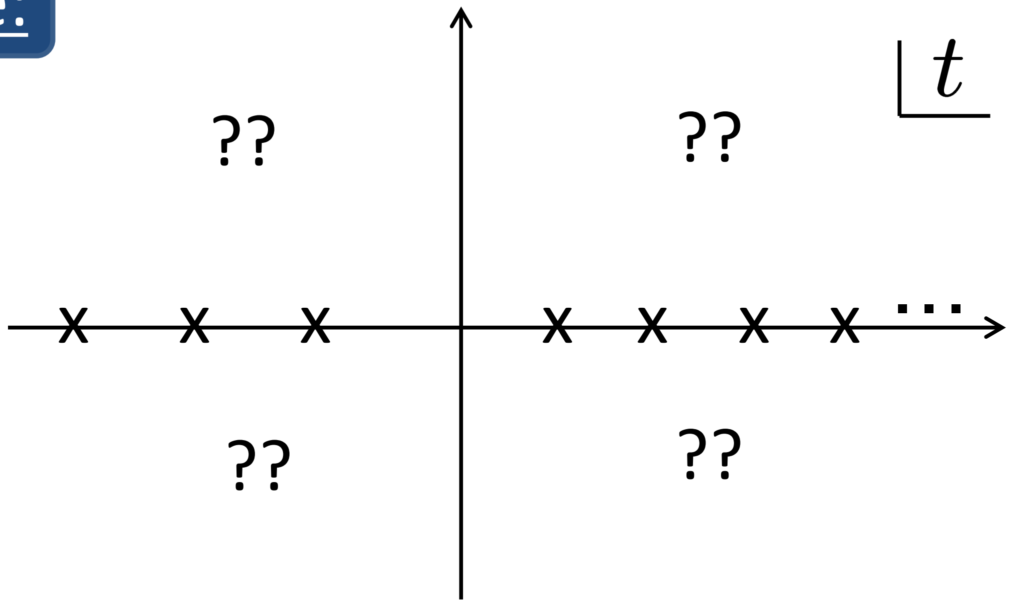
(・ similar for 't Hooft loop, but ∃ monopole bubbling effects) [MH-Yokoyama]

# Expectations in typical QFT

['t Hooft '79]

Non-Borel summable due to singularities along $\boldsymbol{R}_+$

Borel plane:



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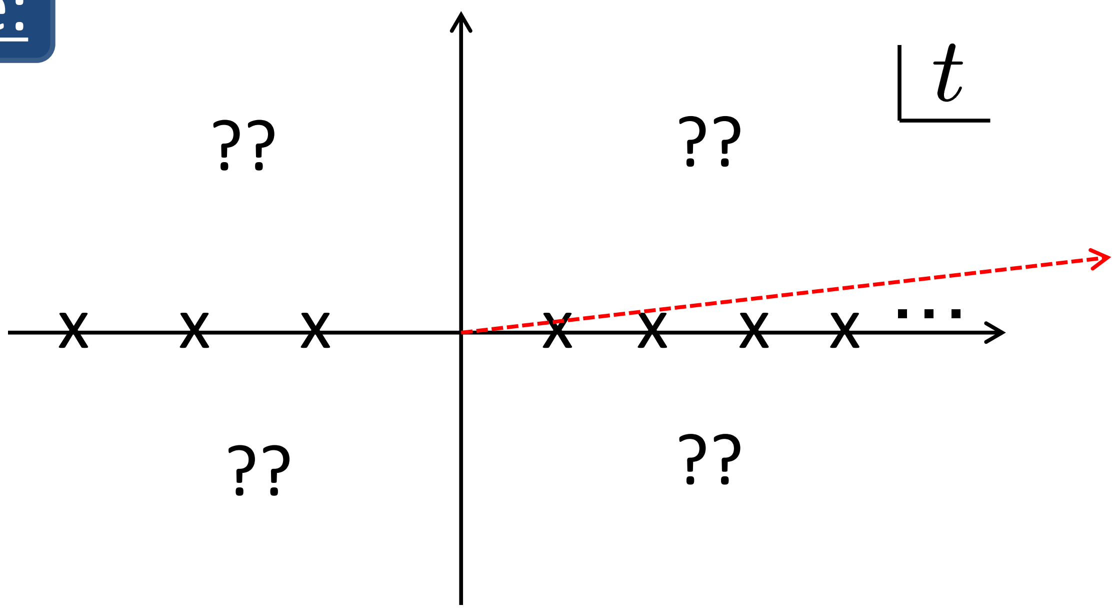

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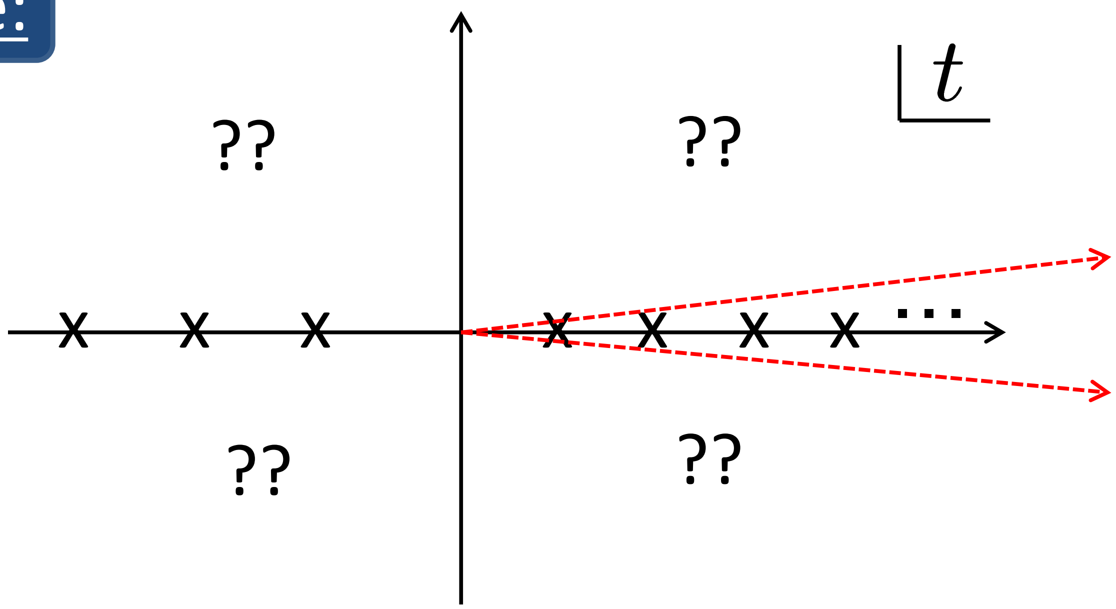

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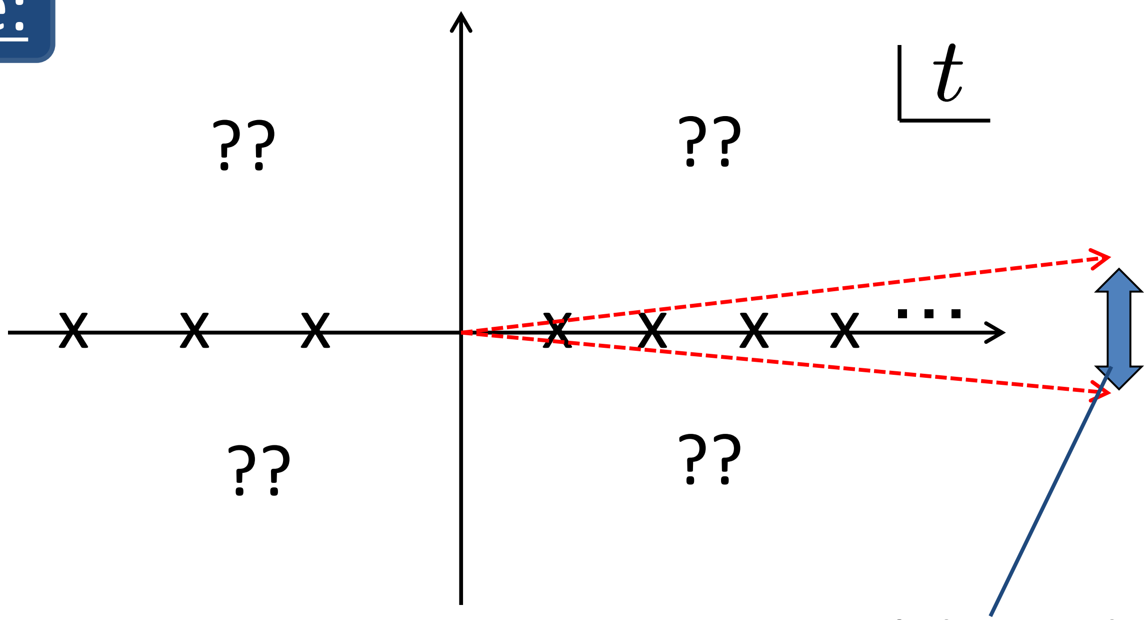


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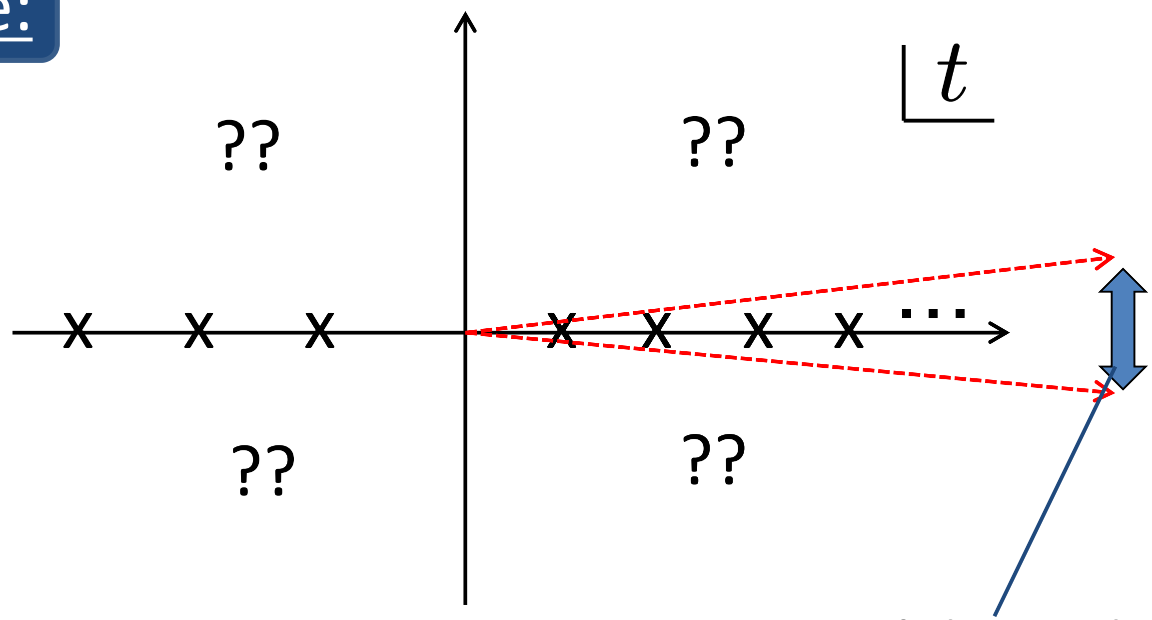


Integral depends on a way to avoid singularities

$$S_{\theta=0}\mathcal{O}(g) = \int_0^\infty dt\ e^{-\frac{t}{g}}\ \mathcal{BO}(t) \Rightarrow (\text{Residue}) \sim e^{-\frac{\sharp}{g}}$$

*Non-perturbative effect?*

# Interpretation of Borel singularities

$$Z(g) = \int D\Phi e^{-\frac{1}{g}S[\Phi]} \simeq \sum_{\ell} c_\ell g^\ell$$

[Lipatov '77]

Large order coefficient:

$$c_\ell = \frac{1}{2\pi i} \oint \frac{dg}{g^{\ell+1}} Z(g) = \frac{1}{2\pi i} \oint dg \int D\phi e^{-\frac{1}{g}S[\phi] - (\ell+1)\ln g} \qquad (\ell \to \infty)$$

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$$\simeq e^{-\frac{1}{g_*}S[\phi_*]-(\ell+1)\ln g_*} \qquad \left( \left.\frac{\delta S}{\delta \phi}\right|_{\phi=\phi_*} = 0, \ -\frac{1}{g_*^2}S[\phi_*] + \frac{\ell+1}{g_*} = 0 \right)$$

$$= e^{(\ell+1)\ln(\ell+1)-(\ell+1)} (S[\phi_*])^{-(\ell+1)} \simeq \ell!\,(S[\phi_*])^{-(\ell+1)}$$

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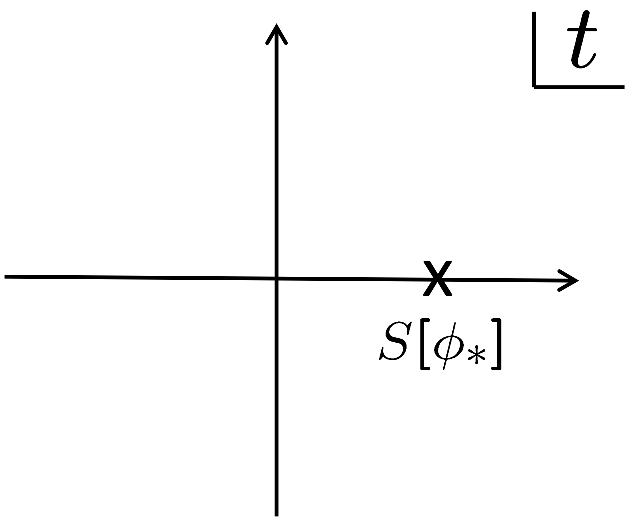
$$= e^{(\ell+1)\ln(\ell+1)-(\ell+1)} \left(S[\phi_*]\right)^{-(\ell+1)} \simeq \ell! \left(S[\phi_*]\right)^{-(\ell+1)}$$

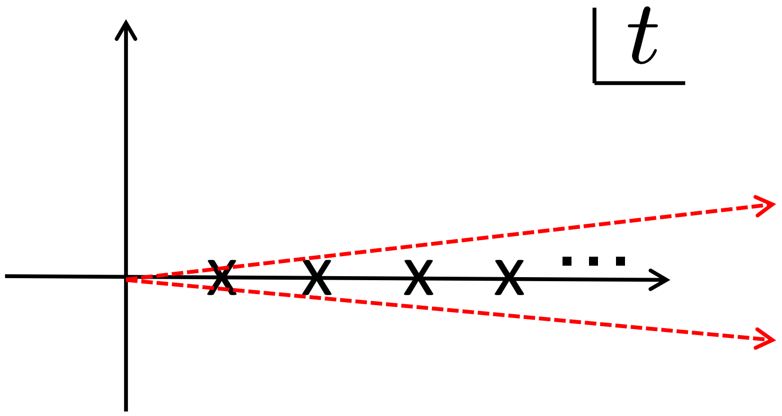
$$\mathcal{B}Z(t) \simeq \sum_\ell \left(S[\phi_*]\right)^{-\ell} = \frac{1}{1-\frac{t}{S[\phi_*]}}$$




Nontrivial saddle point gives Borel singularities

# Contents

# Resurgence

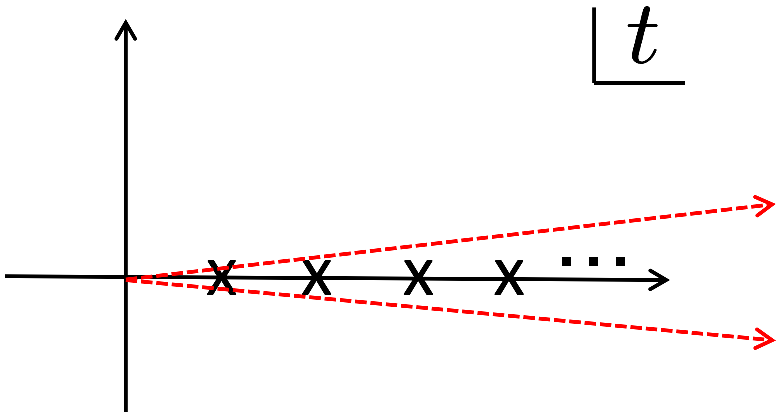



$$(\text{Ambiguities}) \sim (\text{Residue}) \sim e^{-\frac{\sharp}{g}}$$

**Idea of resurgence:** (explicit examples in next slides)

This is precisely canceled by ambiguities of perturbative series around other saddle points (~ non-pert. sector):

# Resurgence

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**Idea of resurgence:** (explicit examples in next slides)

This is precisely canceled by ambiguities of perturbative series around other saddle points (∼ non-pert. sector):

(perturbative ambiguity) = −(non-perturbative ambiguity)

⇒ (unambiguous answer)

# Ex.1: Stirling's formula v.s. Exact gamma function

$$\log n! \sim n \log n$$

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## Improved Stirling's formula:

[cf. Nemes '14]

$$\log \Gamma(z) \sim z \log z - z - \frac{1}{2} \log \frac{z}{2\pi} + I_{\text{pert}}(z) + \sum_{\pm} \sum_{m=1}^{\infty} c_m^{\pm} e^{\pm 2\pi i m z}$$

$$I_{\text{pert}}(z) = \sum_{n=1}^{\infty} \frac{B_{2n}}{2n(2n-1) z^{2n-1}},$$

$$\sim \sum_{n} \frac{(2n)!}{z^{2n-1}}$$

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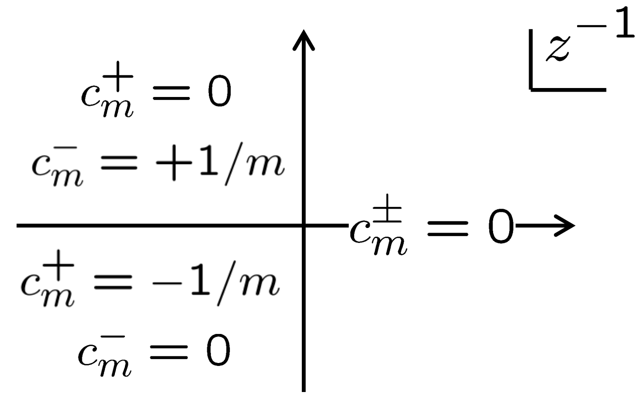


Stokes phenomena!

(Jump of the form of asymptotic expansion)

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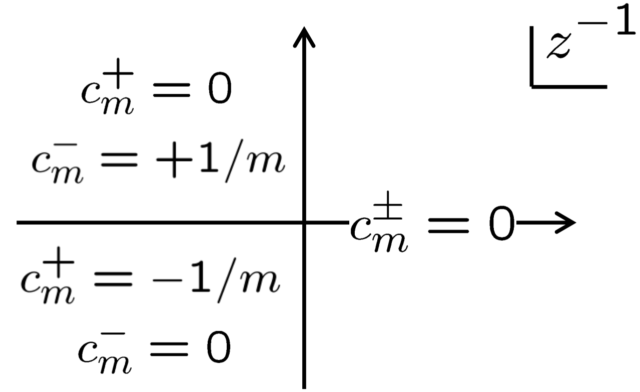



Stokes phenomena!

(Jump of the form of asymptotic expansion)

## Borel resum. in perturbative sector:

$$S_{\arg z^{-1}} I_p(z) = \int_0^{e^{i\arg z^{-1}}\infty} dt\ e^{-zt}\mathcal{B}I_p(t) = \int_0^{e^{i\arg z^{-1}}\infty} dt\ \frac{e^{-zt}}{t}\left[\frac{1}{e^t - 1} - \frac{1}{t} + \frac{1}{2}\right]$$

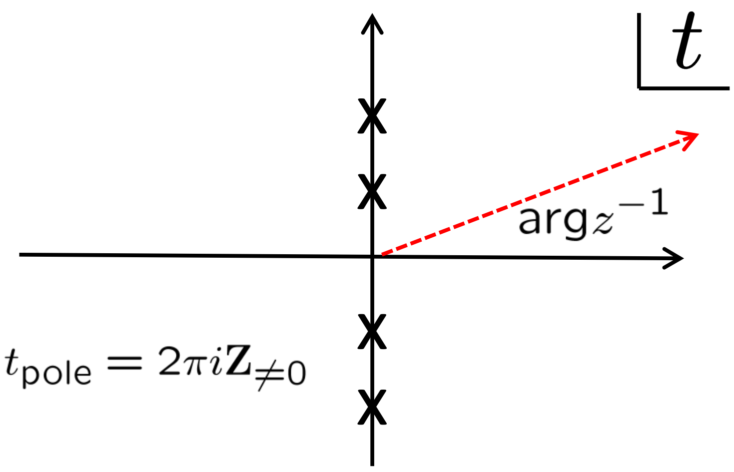
It is known for Re(z)>0,

[Binet's formula]

$$\log\Gamma(z) = z\log z - z - \frac{1}{2}\log\frac{z}{2\pi} + \int_0^{\infty} dt\ \frac{e^{-zt}}{t}\left[\frac{1}{e^t - 1} - \frac{1}{t} + \frac{1}{2}\right]$$

What for Re(z)≦0?

## Perturbative sector:

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Borel ambiguity at $\arg z^{-1} = \pi/2$ :

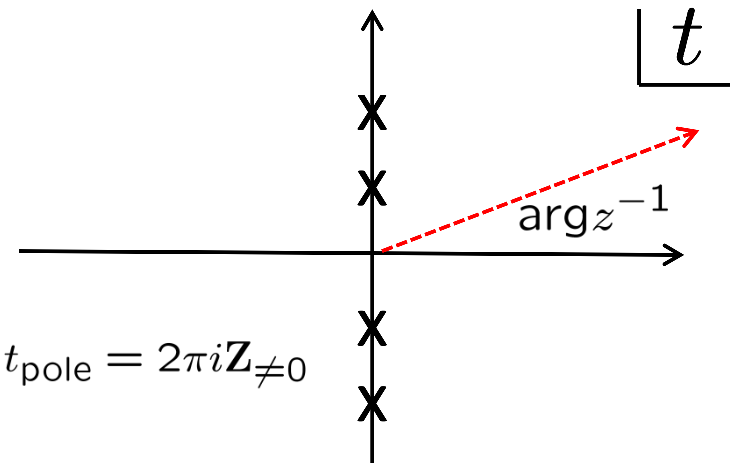
$$(S_{\pi/2+0_+} - S_{\pi/2-0_+}) I_p(z)$$

$$= -\sum_{m=1}^{\infty} \text{Res}_{t=2m\pi i}\left(e^{-zt}\mathcal{B}I_p(t)\right)$$

$$= -\sum_{m=1}^{\infty} \frac{1}{m} e^{-2\pi i m z}$$

## Non-perturbative sector:

## Perturbative sector:

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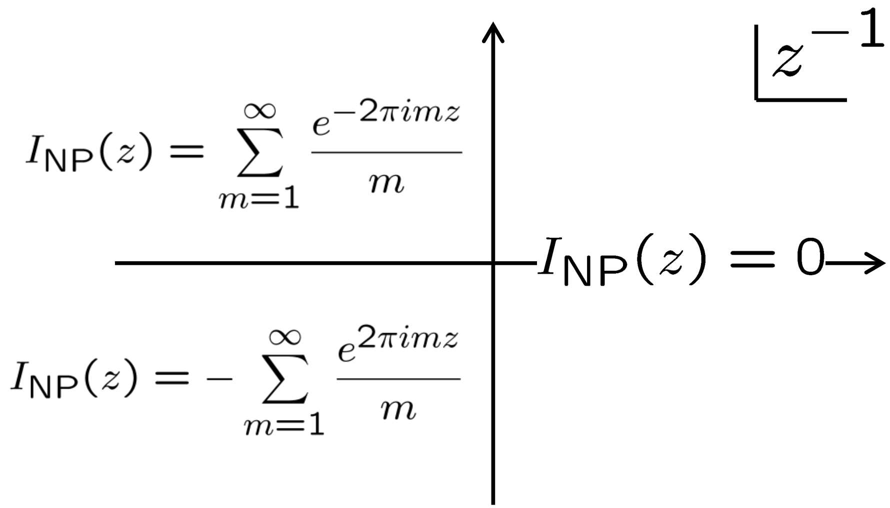
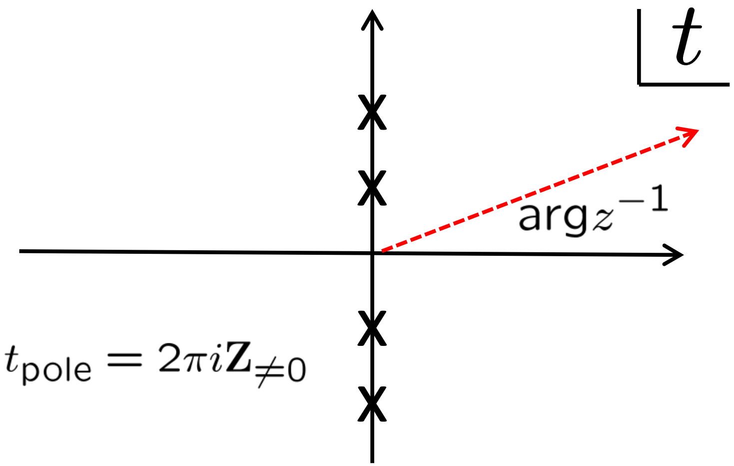



Stokes phenomena generates ambiguities

## Perturbative sector:




$$S_{\arg z^{-1}} I_p(z) = \int_0^{e^{i \arg z^{-1}} \infty} dt \, \frac{e^{-zt}}{t} \left[ \frac{1}{e^t - 1} - \frac{1}{t} + \frac{1}{2} \right]$$

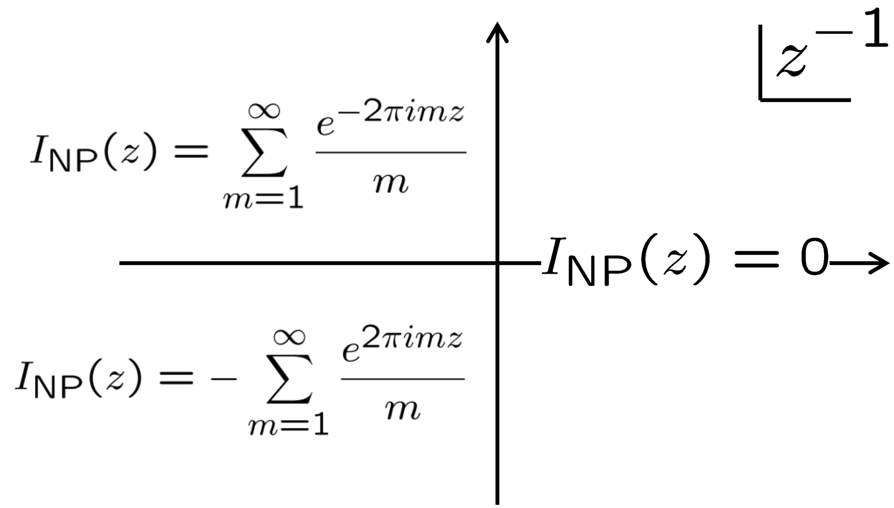
<u>Borel ambiguity at $\arg z^{-1} = \pi/2$ :</u>

$$(S_{\pi/2+0_+} - S_{\pi/2-0_+}) I_p(z)$$

$$= - \sum_{m=1}^{\infty} \text{Res}_{t=2m\pi i} \left( e^{-zt} \mathcal{B} I_p(t) \right)$$

$$= - \sum_{m=1}^{\infty} \frac{1}{m} e^{-2\pi i m z}$$

## Non-perturbative sector:




Stokes phenomena generates ambiguities

<u>Ambiguity at $\arg z^{-1} = \pi/2$ :</u>

$$I_{\text{NP}}(z)|_{\arg z^{-1} = \frac{\pi}{2} + 0_+} - I_{\text{NP}}(z)|_{\arg z^{-1} = \frac{\pi}{2} - 0_+}$$

$$= + \sum_{m=1}^{\infty} \frac{1}{m} e^{-2\pi i m z}$$

**Canceled!** (similar for $\arg z^{-1} = -\pi/2$ )

# An example like QFT

## 0d Sine-Gordon model:

[Cherman-Dorigoni-Unsal '14, Cherman-Koroteev-Unsal '14]

$$Z(g) = \frac{1}{\sqrt{g}} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} dx \; e^{-\frac{1}{2g}\sin^2 x} = \frac{\pi}{\sqrt{g}} e^{-\frac{1}{4g}} I_0\left(\frac{1}{4g}\right)$$

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## Saddle point:

$$0 = \frac{d}{dx}\sin^2 x\Big|_{x=x_*} = \sin(2x_*) \quad \Rightarrow \quad x_* = 0, \ \pm\frac{\pi}{2}$$

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## "Action":

$$\left(S(x) = \frac{1}{2g}\sin^2 x\right)$$

$$S(x=0) = 0$$ *trivial*

$$S\left(x = \pm\frac{\pi}{2}\right) = \frac{1}{2g}$$ *Non-perturbative*

# Expansion around the saddle pts:

$$Z(g) \sim \underbrace{\sum_{\ell=0}^{\infty} c_\ell^{(0)} g^\ell}_{x_* = 0} + \underbrace{e^{-\frac{1}{2g}} \sum_{\ell=0}^{\infty} c_\ell^{(1)} g^\ell}_{x_* = \pm\frac{\pi}{2}} \ ??$$

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## Trivial saddle:

$$Z(g)|_{x_*=0} = \sqrt{2\pi} \sum_{\ell=0}^{\infty} \frac{\Gamma(\ell+1/2)^2 2^\ell}{\Gamma(\ell+1)\Gamma(1/2)^2} g^\ell \equiv \Phi_0(g)$$

$$\Rightarrow \quad \mathcal{B}\Phi_0(t) = \sum_{\ell=0}^{\infty} \frac{c_\ell^{(0)}}{\ell!} t^\ell = \sqrt{2\pi}\ {}_2F_1\left(\frac{1}{2}, \frac{1}{2}, 1; 2t\right)$$

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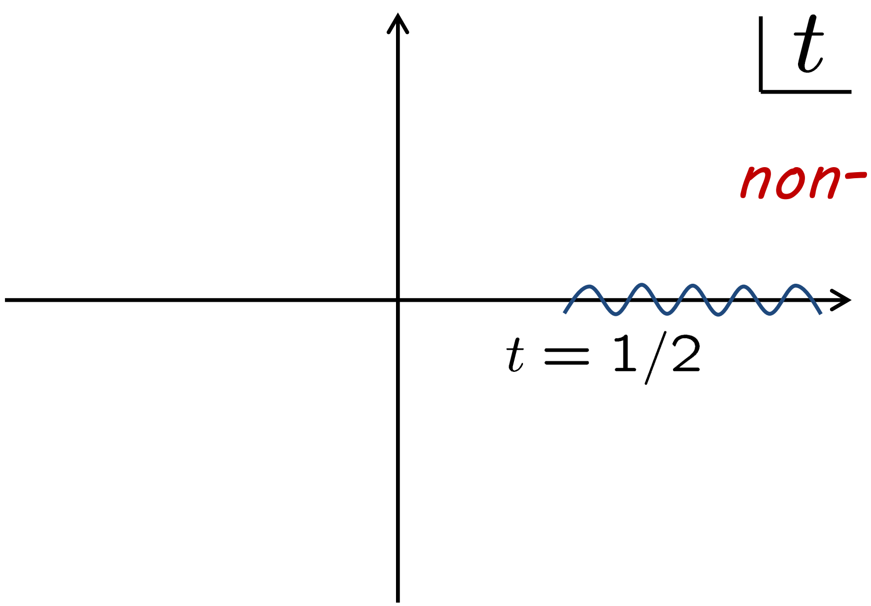
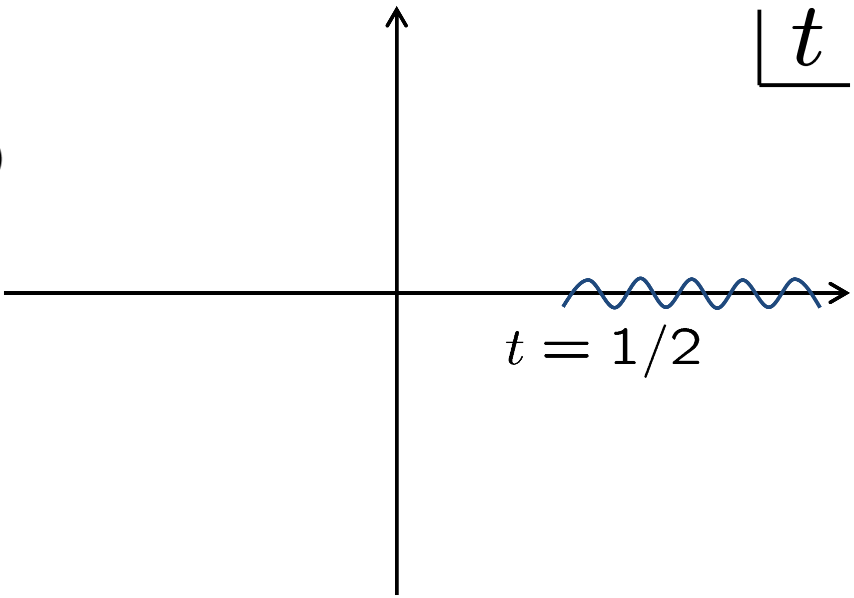



*non-Borel summable!*

$$\begin{cases} S_\theta \Phi_0(g) = \frac{1}{g}\int_0^{e^{i\theta}\infty} dt\ e^{-\frac{t}{g}}\ \mathcal{B}\Phi_0(t) \\ \mathcal{B}\Phi_0(t) = \sqrt{2\pi}\ {}_2F_1\left(\frac{1}{2},\frac{1}{2},1;2t\right) \\ g = |g|e^{i\theta} \end{cases}$$



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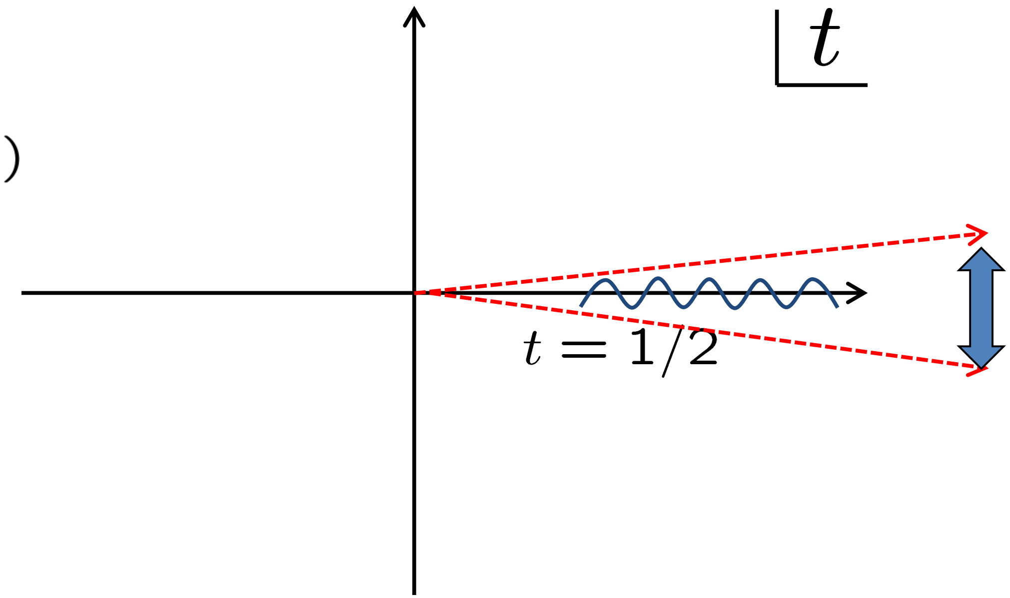

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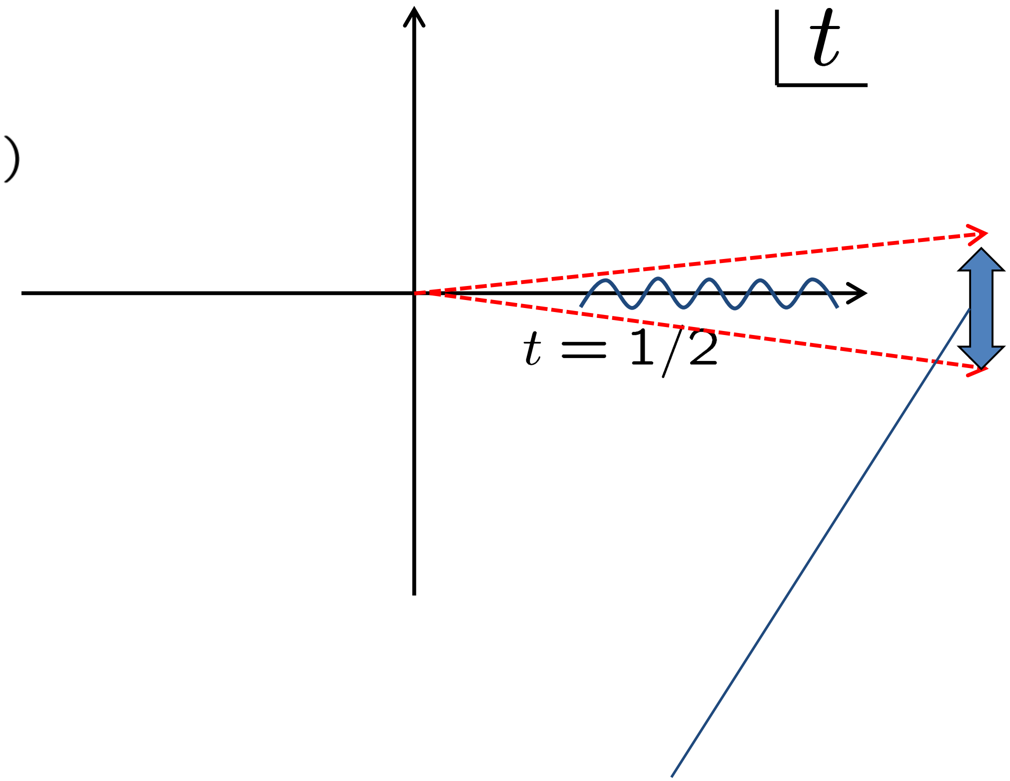


## Ambiguity:

$$\left(S_{0+} - S_{0-}\right)\Phi_0(g) = e^{-\frac{1}{2g}} \times \frac{2i\sqrt{2\pi}}{g}\int_0^{\infty} dt\ e^{-\frac{t}{g}}\ {}_2F_1\left(\frac{1}{2},\frac{1}{2},1;-2t\right) \neq 0$$

Related to contribution from $x_* = \pm\frac{\pi}{2}$ ?

# Expansion around nontrivial saddle

$$\begin{cases} e^{-S(x)} = e^{-\frac{1}{2|g|}e^{-i\theta}x^2+\cdots} & x_* = 0 \\ e^{-S(x)} = e^{-\frac{1}{2g}} \times e^{\frac{1}{2|g|}e^{-i\theta}(x-\pm\frac{\pi}{2})^2+\cdots} & x_* = \pm\frac{\pi}{2} \end{cases} \qquad \left(g = |g|e^{i\theta}\right)$$

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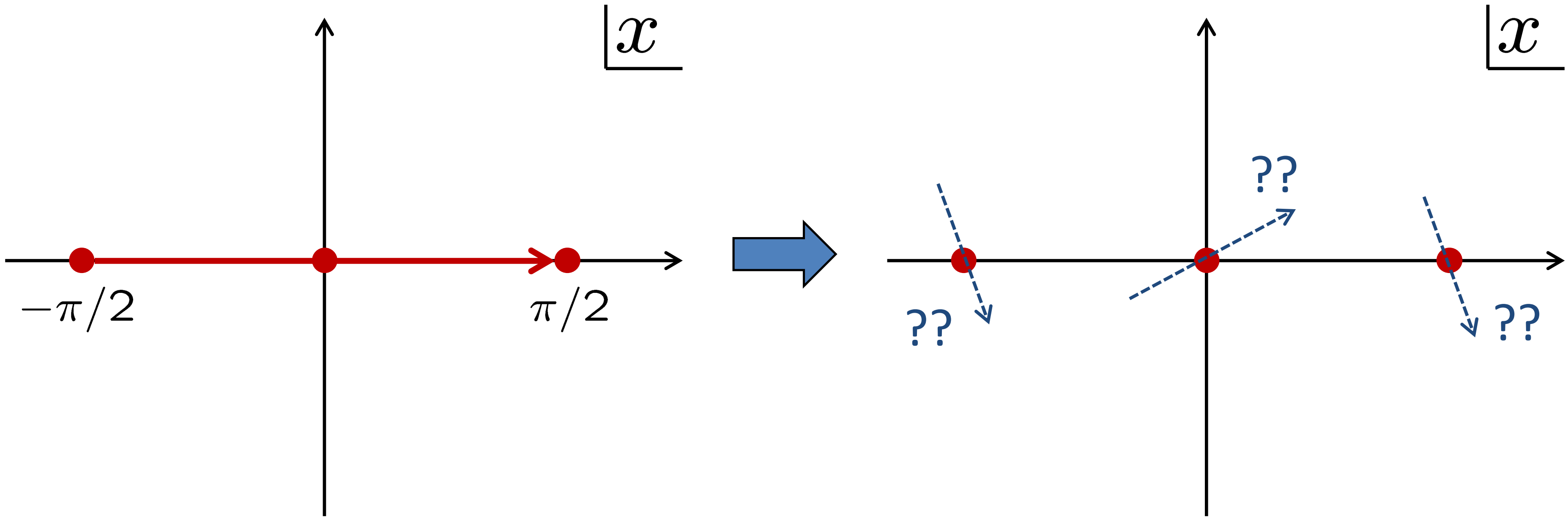
To pick up saddles, change the integral contour to steepest descent s.t.

1. passes the saddles w/ appropriate angle
2. Keep $\mathrm{Im}[S(x)]$ to avoid oscillation
3. Keep the final result (use Cauchy integration theorem)



# Appropriate contour = Lefschetz thimble

[Extension to path integral: Witten '10]

1. Extends real x to complex z
2. Critical pt. : $\left.\frac{dS(z)}{dz}\right|_{z=z_I} = 0$
3. Associated w/ critical pt., ∃ unique Lefschetz thimble $J_I$ :

$$\frac{dz(t)}{dt} = \overline{\frac{\partial S(z)}{\partial z}}, \quad \text{with } z(t \to -\infty) = z_I$$

**Properties:**

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## Properties:

$a)$ $\text{Im}S(z)|_{J_I} = \text{Im}S(z_I)$ $\left(\frac{d}{dt}\text{Im}S \propto \frac{d}{dt}(S-\bar{S}) = \frac{dz}{dt}\frac{\partial S}{\partial z} - \frac{d\bar{z}}{dt}\overline{\frac{\partial S}{\partial z}} = 0\right)$

$b)$ $\text{Re}S(z)|_{J_I} \geq \text{Re}S(z_I)$ $\left(\frac{d}{dt}\text{Re}S \propto \frac{dz}{dt}\frac{\partial S}{\partial z} + \frac{d\bar{z}}{dt}\overline{\frac{\partial S}{\partial z}} = 2\frac{\partial S}{\partial z}\overline{\frac{\partial S}{\partial z}} \geq 0\right)$

$c)$ Decomposition of cycle: (if we are not on Stokes line)

$$\int_C = \sum_{I\in\text{saddle}} n_I \int_{J_I} \qquad (n_I \in \mathbf{Z})$$

may jump as changing parameters

# Appropriate contour = Lefschetz thimble

# Dual thimble = steepest ascent

[Extension to path integral: Witten '10]

1. Extends real x to complex z
2. Critical pt. : $\left.\frac{dS(z)}{dz}\right|_{z=z_I} = 0$
3. Associated w/ critical pt., $\exists$ unique dual thimble $K_I$ :

$$\frac{dz(t)}{dt} = -\overline{\frac{\partial S(z)}{\partial z}}, \text{ with } z(t\to -\infty) = z_I$$

**Properties:**

$a)$ $\text{Im}S(z)|_{K_I} = \text{Im}S(z_I)$

$b)$ $\text{Re}S(z)|_{K_I} \leq \text{Re}S(z_I)$

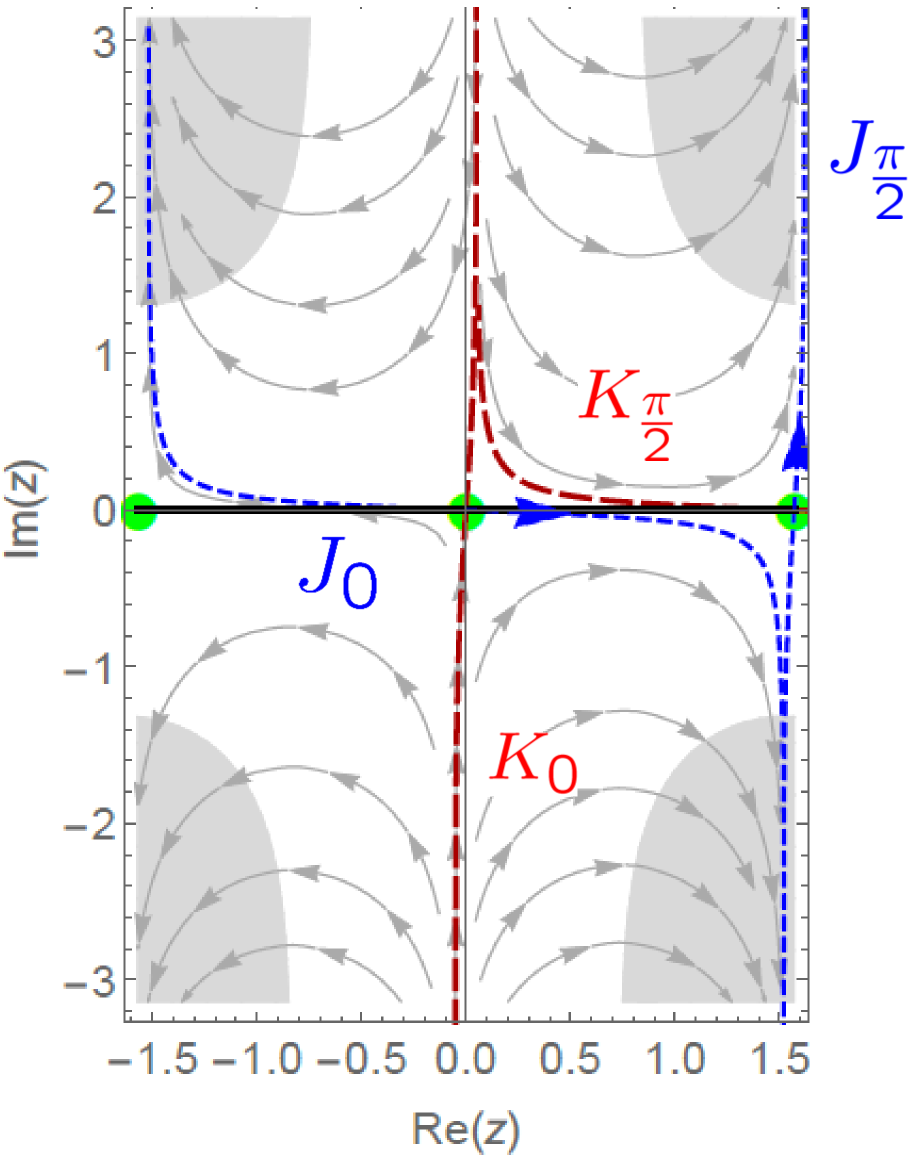
$c)$ Decomposition of cycle: (if we are not on Stokes line)

$$\int_C = \sum_{I\in\text{saddle}} n_I \int_{J_I}, \quad n_I = \text{intersection } \sharp \text{ of } (C, K_I)$$

# Thimble structures in the toy model

[similar to fig.1 in Cherman-Dorigoni-Unsal '14]

$\arg(g) = -0.1$



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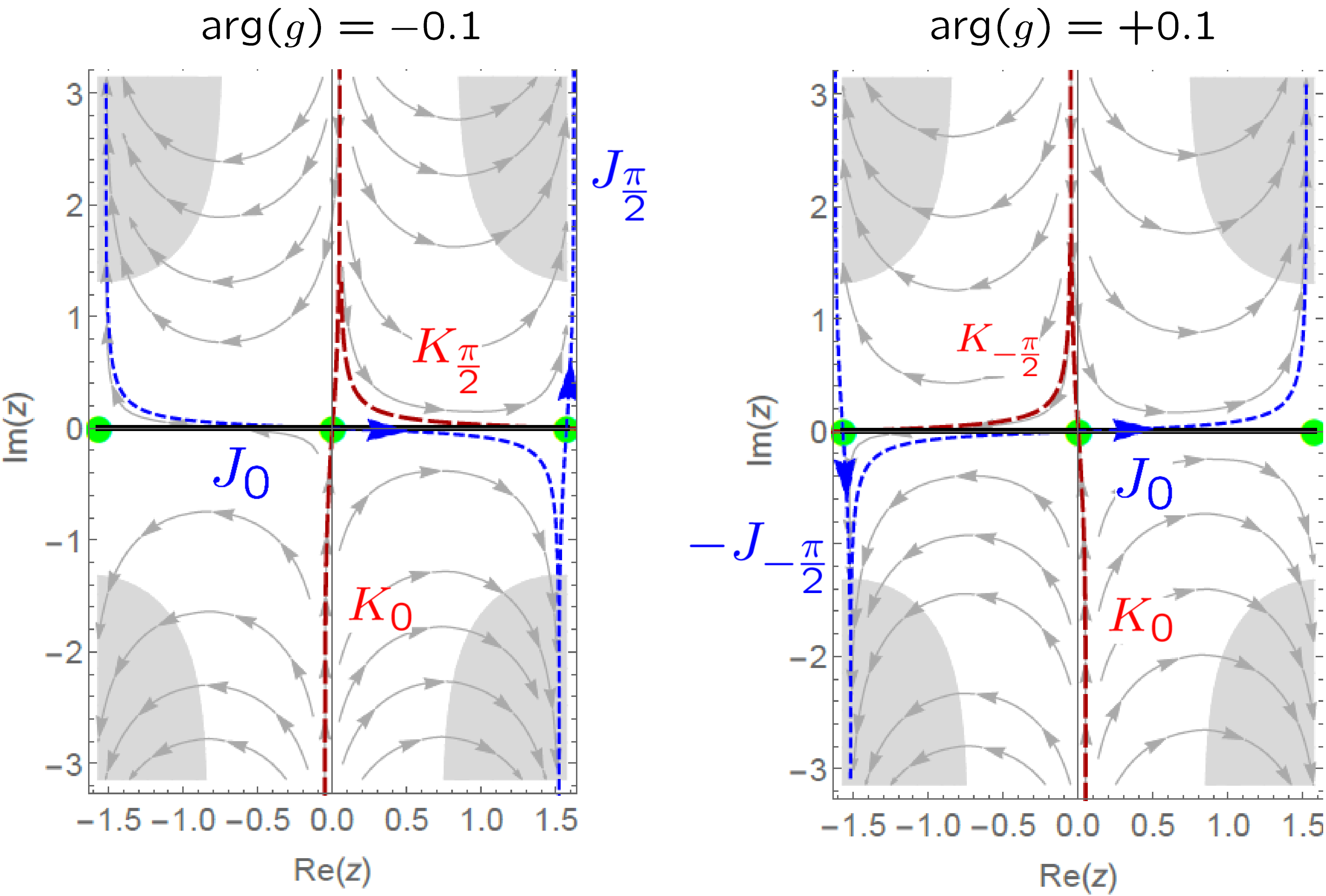

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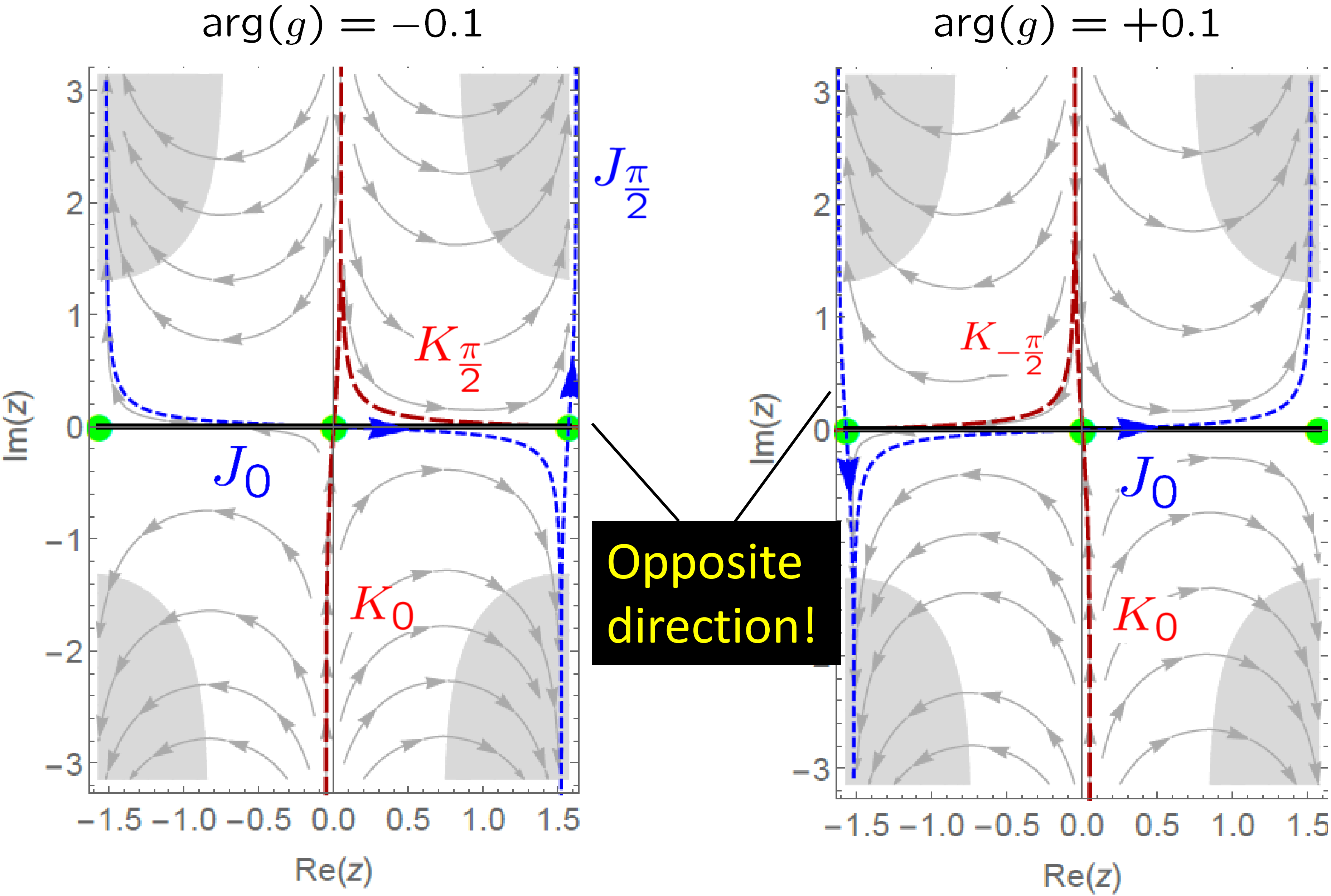
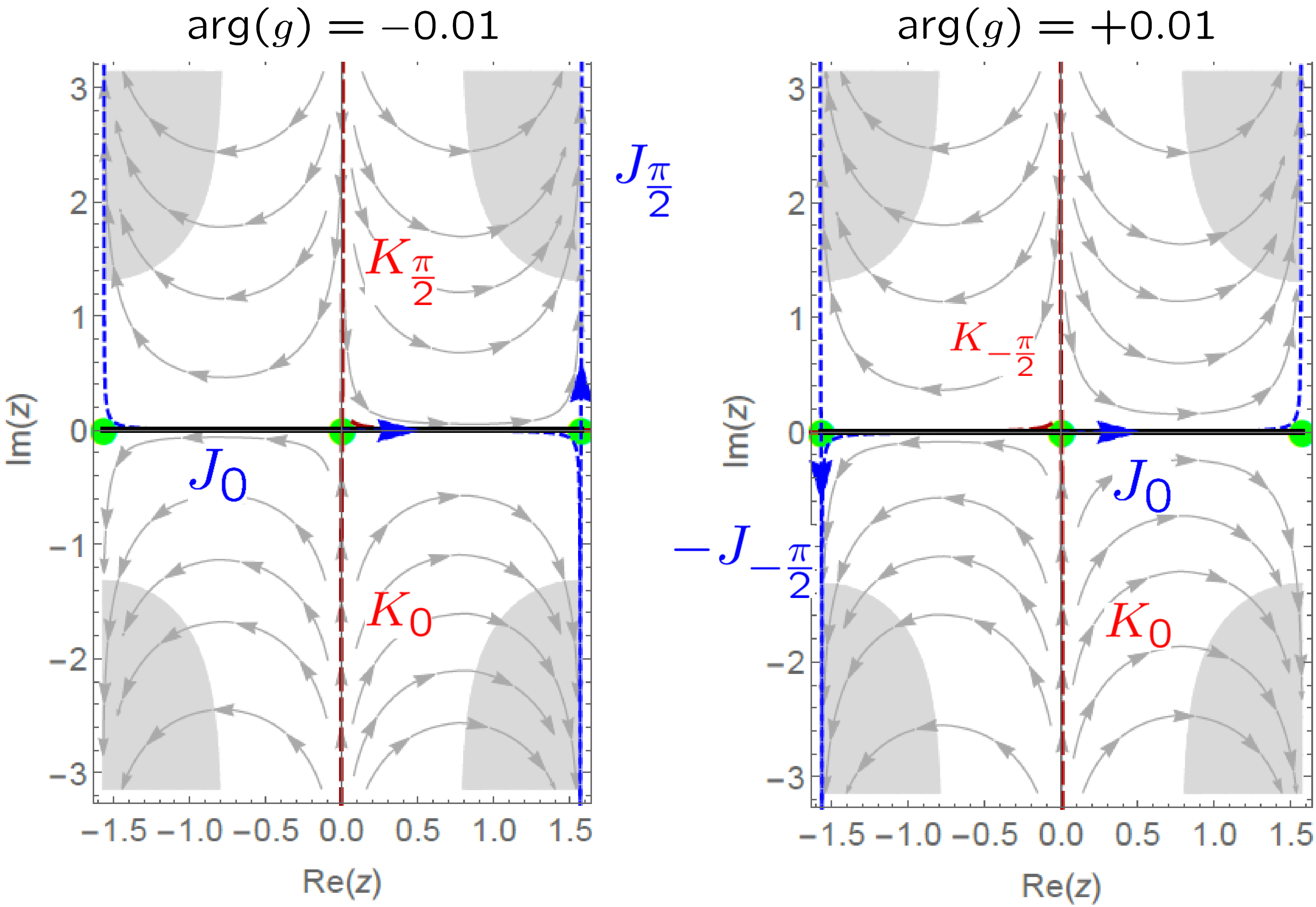
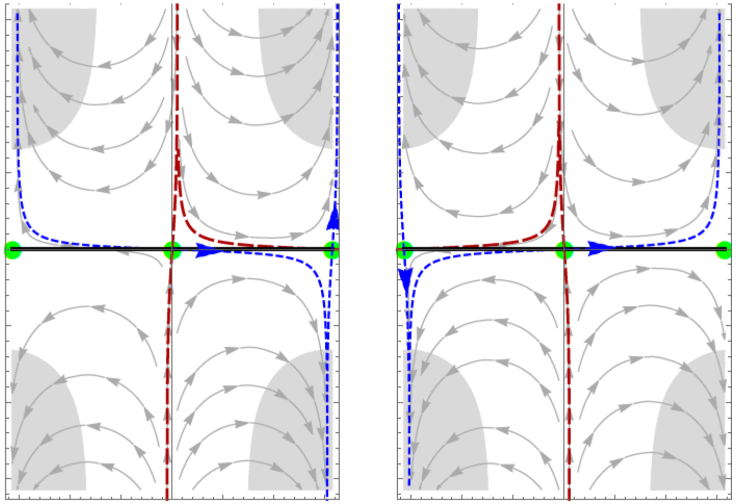


# Thimble structures in the toy model (Cont'd)



# Contribution from nontrivial saddle

- Either $x = +\frac{\pi}{2}$ or $-\frac{\pi}{2}$ contributes
- Contours smoothly change in the ranges $0 < \theta < \pi$ & $-\pi < \theta < 0$
- Contours through nontrivial saddles are opposite between $\theta < 0$ & $\theta > 0$



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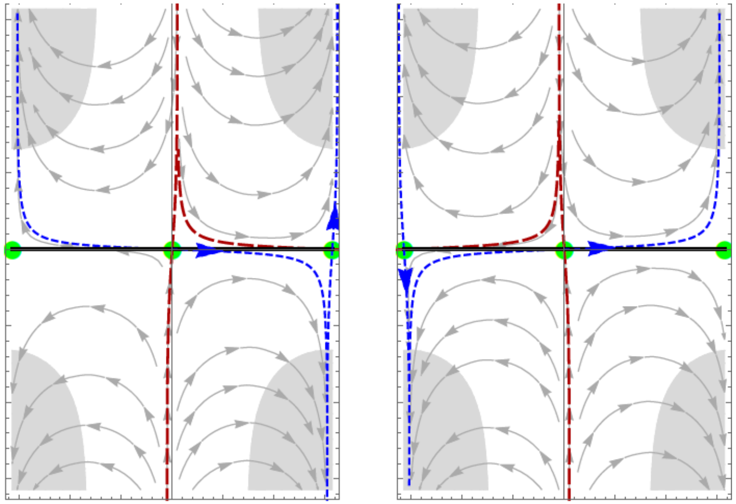

$$Z(g)|_{x_*=\pm\frac{\pi}{2}} = \begin{cases} +e^{-\frac{1}{2g}} \sum_{\ell=0}^{\infty} c_\ell^{(1)} g^\ell & (\theta < 0) \\ -e^{-\frac{1}{2g}} \sum_{\ell=0}^{\infty} c_\ell^{(1)} g^\ell & (\theta > 0) \end{cases}$$

$^\exists$Jump at $\theta = 0$ !! ("Stokes phenomenon")

Expansion around nontrivial saddle is also ambiguous at $\theta = 0$

# Expansion around nontrivial saddle

$$\pm e^{-\frac{1}{2g}}\sqrt{2\pi}\sum_{\ell=0}^{\infty}\frac{(-2)^{\ell}\Gamma(\ell+1/2)^2}{\Gamma(\ell+1)\Gamma(1/2)^2}g^{\ell} \equiv \pm e^{-\frac{1}{2g}}\Phi_1(g)$$

$$\Rightarrow \quad \mathcal{B}\Phi_1(t) = \sum_{\ell=0}^{\infty}\frac{c_{\ell}^{(1)}}{\ell!}t^{\ell} = \sqrt{2\pi}\ {}_2F_1\left(\frac{1}{2},\frac{1}{2},1;-2t\right)$$

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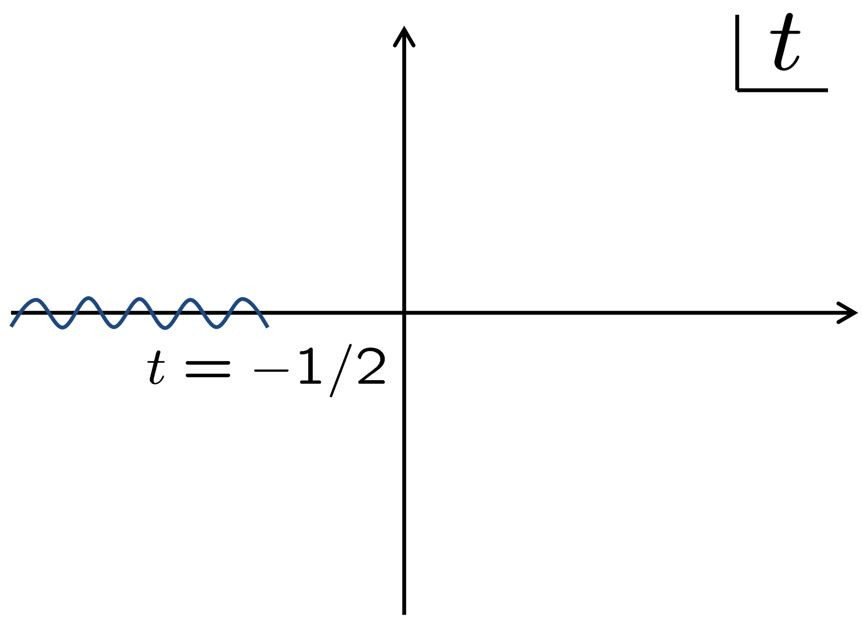
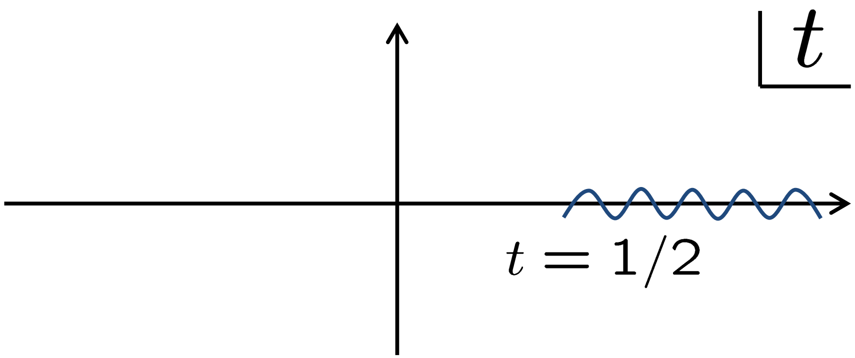



Borel trans. itself is OK but $^{\exists}$ambiguity at $\theta=0$ because of Stokes phenomena

# Comparison of ambiguities (at θ=0)

## Trivial saddle




By the branch cut, ambiguity:

$$\left(S_{0+} - S_{0-}\right)\Phi_0(g)$$
$$= e^{-\frac{1}{2g}} \frac{2i\sqrt{2\pi}}{g} \int_0^\infty dt\ e^{-\frac{t}{g}}\ {}_2F_1\left(\frac{1}{2}, \frac{1}{2}, 1; -2t\right)$$

## Nontrivial saddle

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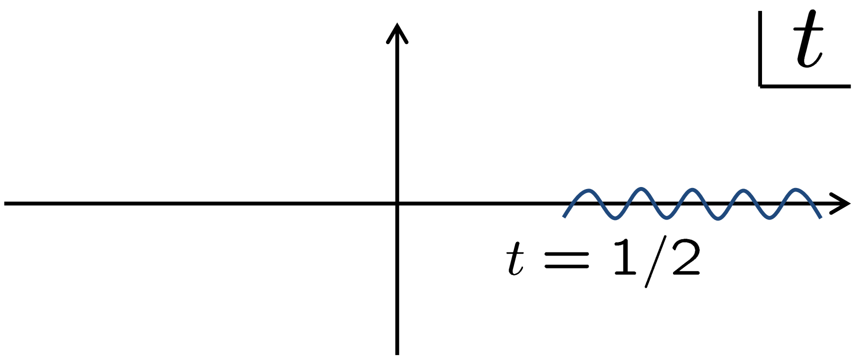


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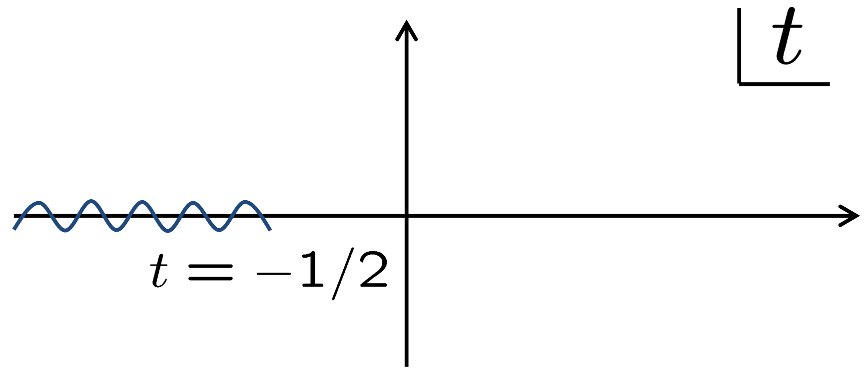

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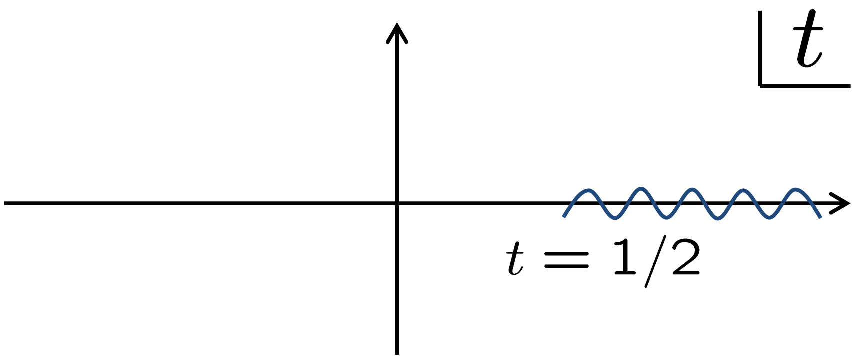


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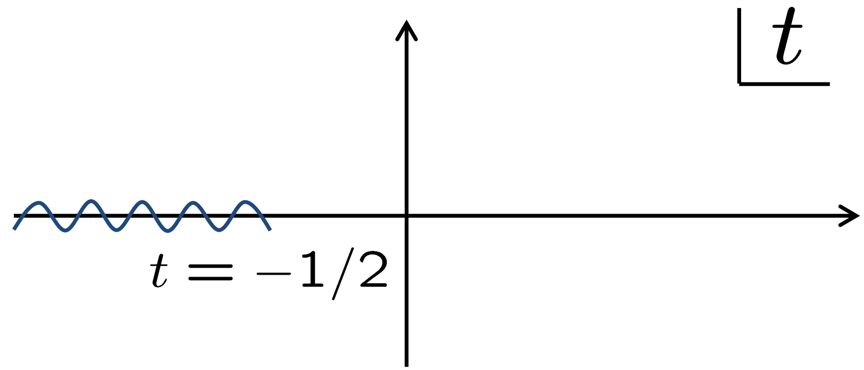


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$$Z(g)|_{x_* = \pm\frac{\pi}{2}} = \begin{cases} +ie^{-\frac{1}{2g}} S_\theta \Phi_1(g) & (\theta < 0) \\ -ie^{-\frac{1}{2g}} S_\theta \Phi_1(g) & (\theta > 0) \end{cases}$$

Ambiguity:

$$-2ie^{-\frac{1}{2g}} S_0 \Phi_1(g) = -\frac{2i\sqrt{2\pi}}{g} e^{-\frac{1}{2g}} \int_0^\infty dt\ e^{-\frac{t}{g}}\ {}_2F_1\left(\frac{1}{2}, \frac{1}{2}, 1; -2t\right)$$

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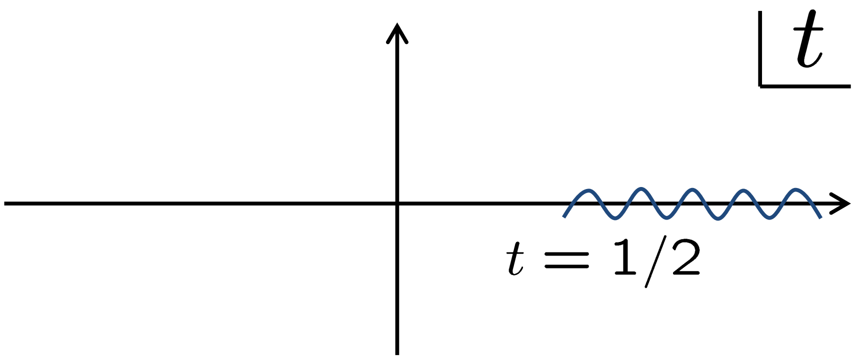


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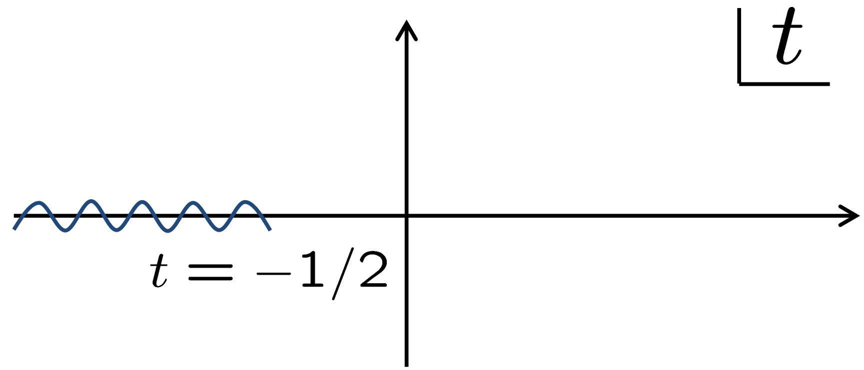


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$$= -\left(S_{0+} - S_{0-}\right)\Phi_0(g)$$

# Resurgence

(Ambiguity from trivial saddle point)

$$||$$

$-$(Ambiguity from nontrivial saddle point)

Resummation from a saddle point may be ambiguous but the ambiguity is cancelled by other saddles

In the toy model, resurgence gives the exact result:

$$Z(g \in \mathbf{R}_{\geq 0}) = \lim_{\theta \to 0_\pm} \left[ S_\theta \Phi_0(g) \mp i e^{-\frac{1}{2g}} S_\theta \Phi_1(g) \right] = \mathrm{Re} S_0 \Phi_0(g)$$

natural to ask if resurgence can be applied to various physics

# Remark 1/4: perturbative ↔ non-perturbative

Ambiguity cancellation:

$$(S_{0+} - S_{0-})\Phi_0(g) = 2ie^{-\frac{1}{2g}} S_0 \Phi_1(g)$$

➡ Relation between perturbative coefficients around trivial & nontrivial saddles

Note: Many talks on resurgence by physicists emphasize this point.

Then some physicists have an impression that definition of resurgence is relations between perturbative and non-perturbative sectors.

If there are ambiguities, there should be cancellations of them but if not, such relations do not have to exist.

Ex.) Ground state energy in system w/ SUSY breaking by non-perturbative effects, Seiberg-Witten prepotential, SUSY obs. in 4d N=2 & 5d N=1 theories on sphere [MH '16]

[Some deformations have nontrivial structures: Dunne-Unsal , Kozcaz-Sulejmanpasic-Tanizaki-Unsal, Dorigoni-Glass ]

# Remark 2/4: The toy model is useful but very special

- We can compute all order perturbative coefficients

  —— In realistic QFT, computing higher order itself deserves to write a paper

- ∃ only one nontrivial saddle points

  —— ∃ $\infty$ many saddles in QFT

- Perturbative series in all the sectors are related

  —— Resurgence doesn't relate different topological sectors

- We can explicitly draw thimbles

  —— impossible in more than two dim. integral

- Perturbative sector knows everything: $Z(g) = \mathrm{Re} S_0 \Phi_0(g)$

  —— not true in more complicated cases

# Remark 3/4: A "Mathematical" viewpoint

## Resurgence $\sim$ "Extension" of analyticity

Analytic function:

$$f(z) = \begin{cases} \sum_n f_n z^n, & |z| < \text{radius of convergence} \\ \text{(analytic continuation)} & \text{everywhere} \end{cases}$$

$\longrightarrow \{1,\ z,\ z^2,\ \cdots\}$ are "good basis" to express f(z)

# Remark 3/4: A “Mathematical” viewpoint

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$\longrightarrow \{1,\ z,\ z^2,\ \cdots\}$ are “good basis” to express f(z)

For more general function, we need more “basis”:

$$\{z^\sharp,\ z^\sharp \log z,\ z^\sharp e^{-\frac{\sharp}{z}},\ \cdots\}$$

Ex.) The toy example needed $\{g^n,\ g^n e^{-\frac{1}{2g}}\}$

# Remark 4/4: Finite order approximation

$$\mathcal{B}\mathcal{O}(t) = \sum_{\ell=0}^{\infty} \frac{c_\ell}{\Gamma(a+\ell)} t^{a+\ell-1}$$

To compute Borel trans.,

we need all order perturbative coefficients in principle.

But when we know only up to finite order,

# Remark 4/4: Finite order approximation

$$\mathcal{BO}(t) = \sum_{\ell=0}^{\infty} \frac{c_\ell}{\Gamma(a+\ell)} t^{a+\ell-1}$$

To compute Borel trans.,

we need all order perturbative coefficients in principle.

But when we know only up to finite order,

we can use Pade approximation for Borel trans.:

("Borel-Pade approximation")

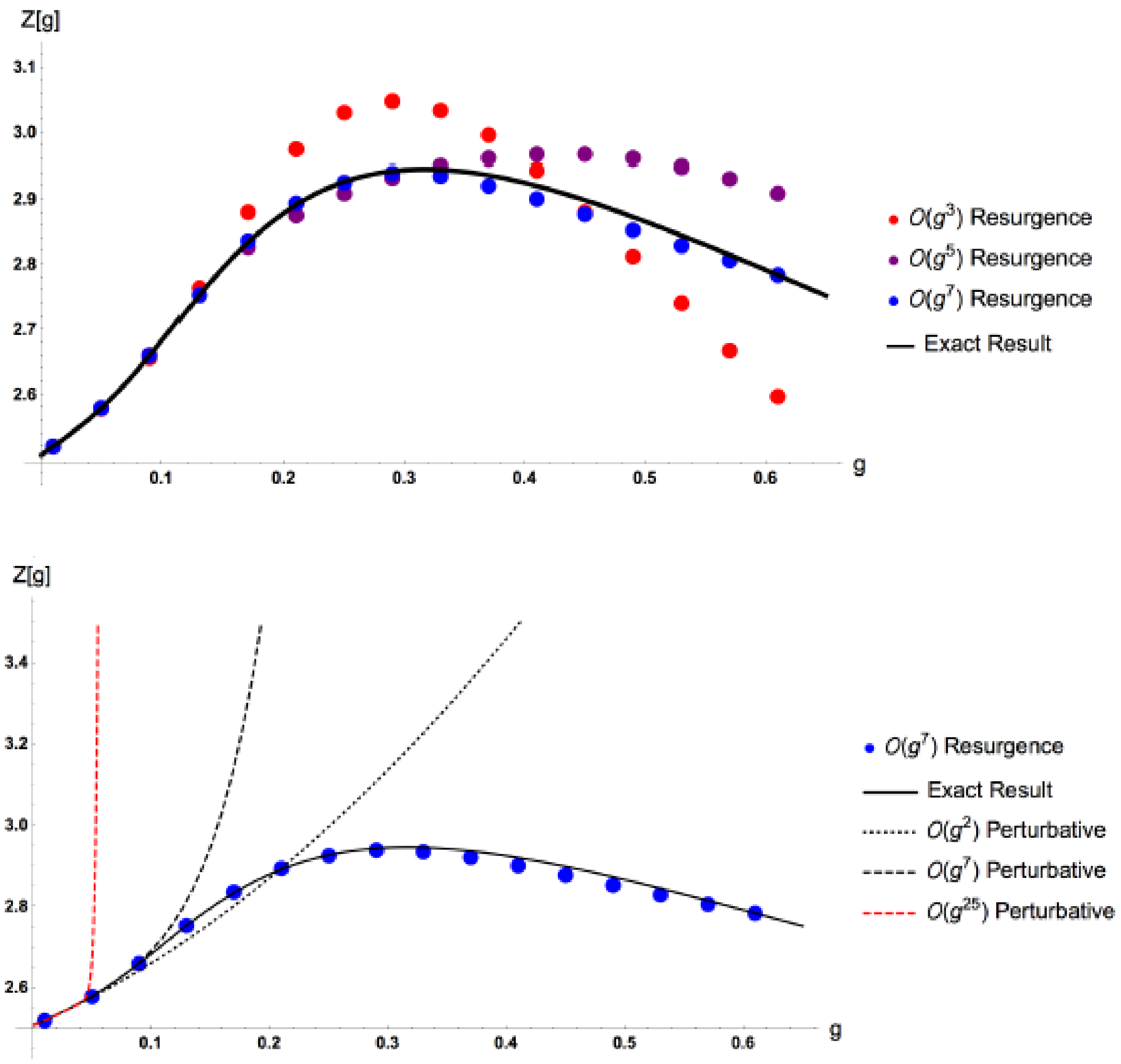
$$P_{m,n}(t) = \frac{\sum_{k=0}^{m} c_k t^k}{1 + \sum_{\ell=1}^{n} d_\ell t^\ell}$$

where coefficients are determined s.t.
small-t expansion gives the one of Borel trans.

# Remark 4/4: Finite order approximation (Cont'd)

## Result in the toy model:

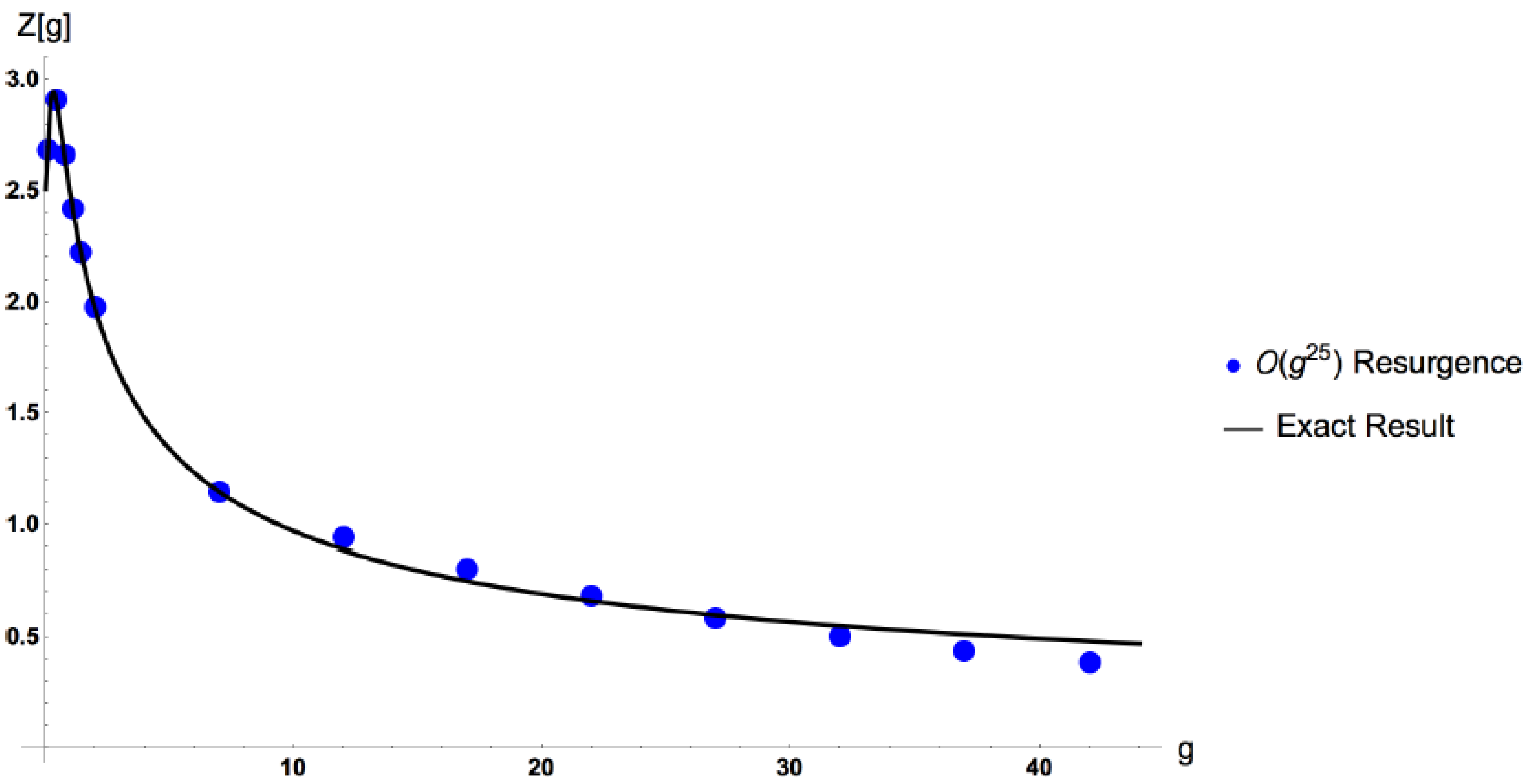
[Fig.4 in Cherman-Koroteev-Unsal '14]



# Remark 4/4: Finite order approximation (Cont'd)

[Fig.5 in Cherman-Koroteev-Unsal '14]

## Result in the toy model:



# Summary

# Summary

- Non-convergent series is ubiquitous in theoretical physics
- Borel singularities $\leftrightarrow$ Nontrivial saddle points
- At first sight, Borel resummation seems usually dead
  & ambiguous due to singularities along $\boldsymbol{R}_+$
- But it may be resurgent.
  The ambiguities from a saddle pt. may be cancelled
  by other saddles
- We should rewrite (path) int. in terms of Lefschetz thimble

# Successful examples of resurgence

## Quantum mechanics

- Quartic/Periodic potential etc. [Zinn Justin-Jentschura '04, etc.]
- $CP^N$ [Fujimori-Kamata-Misumi-Nitta-Sakai '17]
- Slightly broken SUSY [Dunne-Unsal , Kozcaz-Sulejmanpasic-Tanizaki-Unsal '16]

## 2d QFT

- $CP^N$ /O(N) sigma model [Dunne-Unsal '12, Misumi-Nitta-Sakai, etc..] [Dunne-Unsal '15]
- Principal chiral model [Cherman-Dorigoni-Unsal '15]
- Pure Yang-Mills [Ahmed-Dunne '17, Okuyama-Sakai '18]

## 3d QFT

- Pure Chern-Simons [Gukov-Marino-Putrov '16]
- N=2 SUSY Chern-Simons matter theories [MH '16, Gukov-Pei-Putrov-Vafa '17, Fujimori-MH-Kamata-Misumi-Sakai '18]

## Gravity/string

- string theory [Marino-Schiappa, Grassi-Marino-Zakany, Kuroki-Sugino, etc.]
- Quantum cosmology [MH-Matsui-Okabayashi-Terada '24]

Thanks!

# Appendix

# How general are QFT observables as functions of couplings?

~ What are "sufficient basis" to express QFT observables?

**General function**

$$\{g^\sharp,\ g^\sharp \log g,\ g^\sharp e^{-\frac{1}{g^\sharp}},\ g^\sharp e^{-\sharp e^{\frac{\sharp}{g}}}\}$$

QCD? [Aitken-Cherman-Poppitz-Yaffe '17]

[cf. similar suspection in 1/N-expansion of SYK model :
Cotler-Gur-Ari-Hanada-Polchinski-Saad-Shenker-Stanford-Streicher-Tezuka ]

$$\{g^\sharp,\ g^\sharp \log g,\ g^\sharp e^{-\frac{\sharp}{g}}\ \}$$

Successful examples of resurgence so far

**Trivial (No apparent ambiguities)**

SUSY obs. in 4d N=2, 5d N=1 on $S^d$

[MH '16]

**Analytic function** $\{g^n\}$

CFT in 't Hooft limit