Asymptotics in Astrophysics iTHEMS Workshop, Feb. 26 (Wed) 2025

Introduction to black hole quasinormal modes

Naritaka Oshita

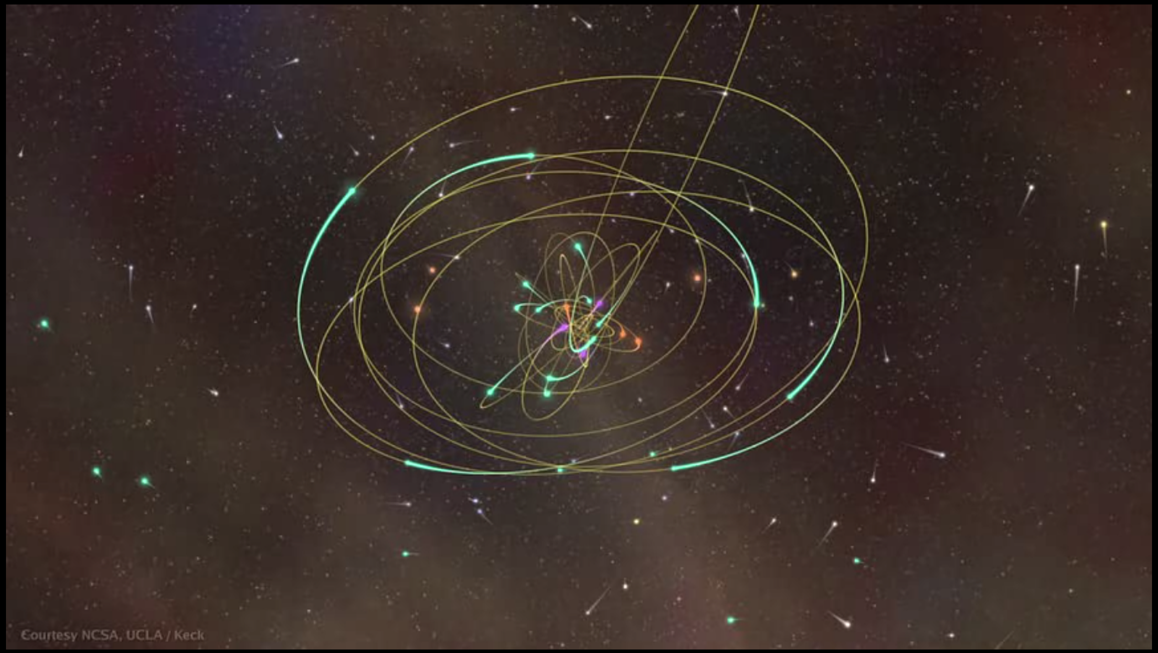
(Kyoto University, RIKEN iTHEMS)

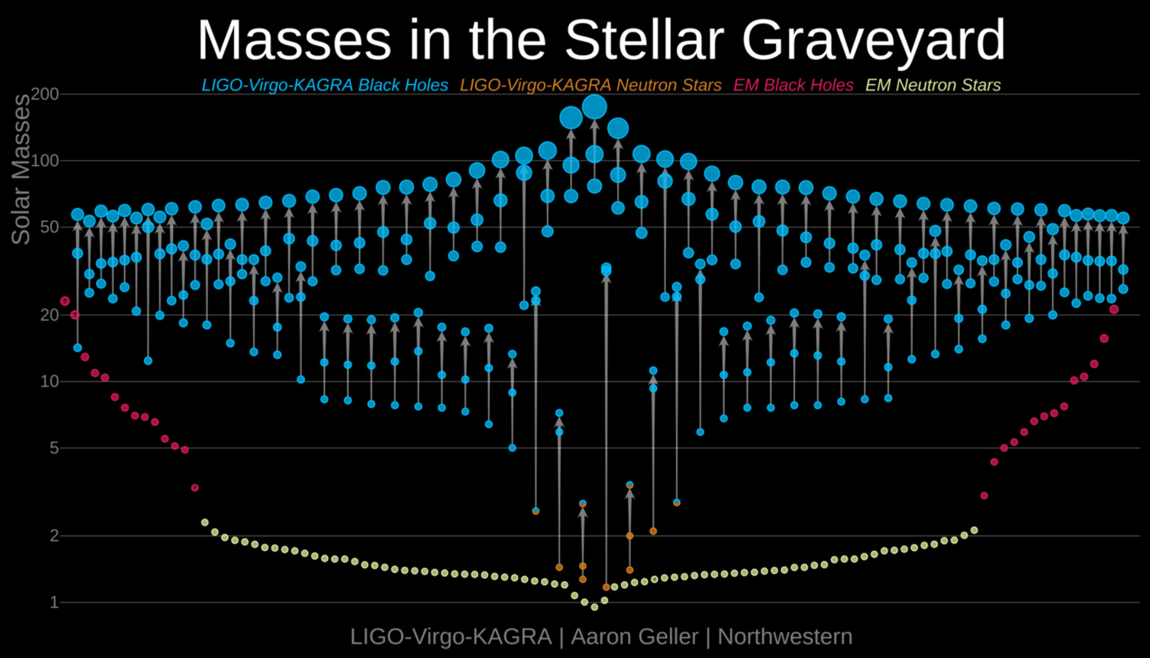
YUKAWA INSTITUTE FOR THEORETICAL PHYSICS

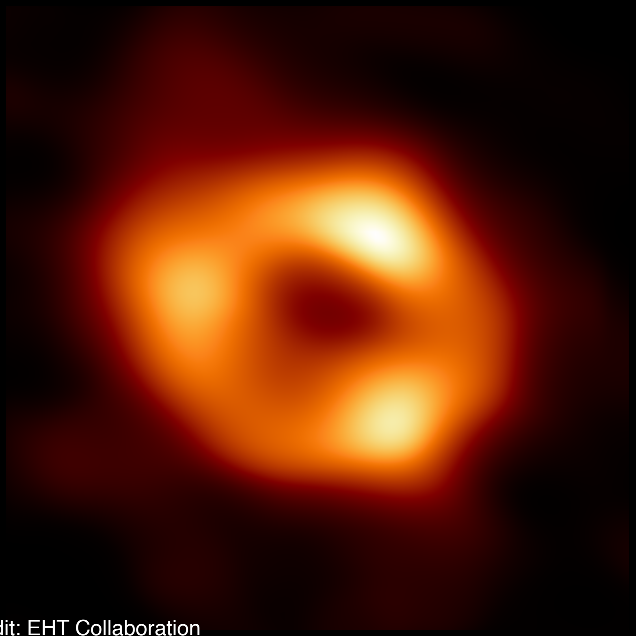
Hakubi Kyoto Univ.

RIKEN Interdisciplinary Theoretical and Mathematical Sciences Program

Discovered Black Holes







Credit: EHT Collaboration

2017

2020

Predicted Black Holes

Singularity theorem by Roger Penrose (1965)

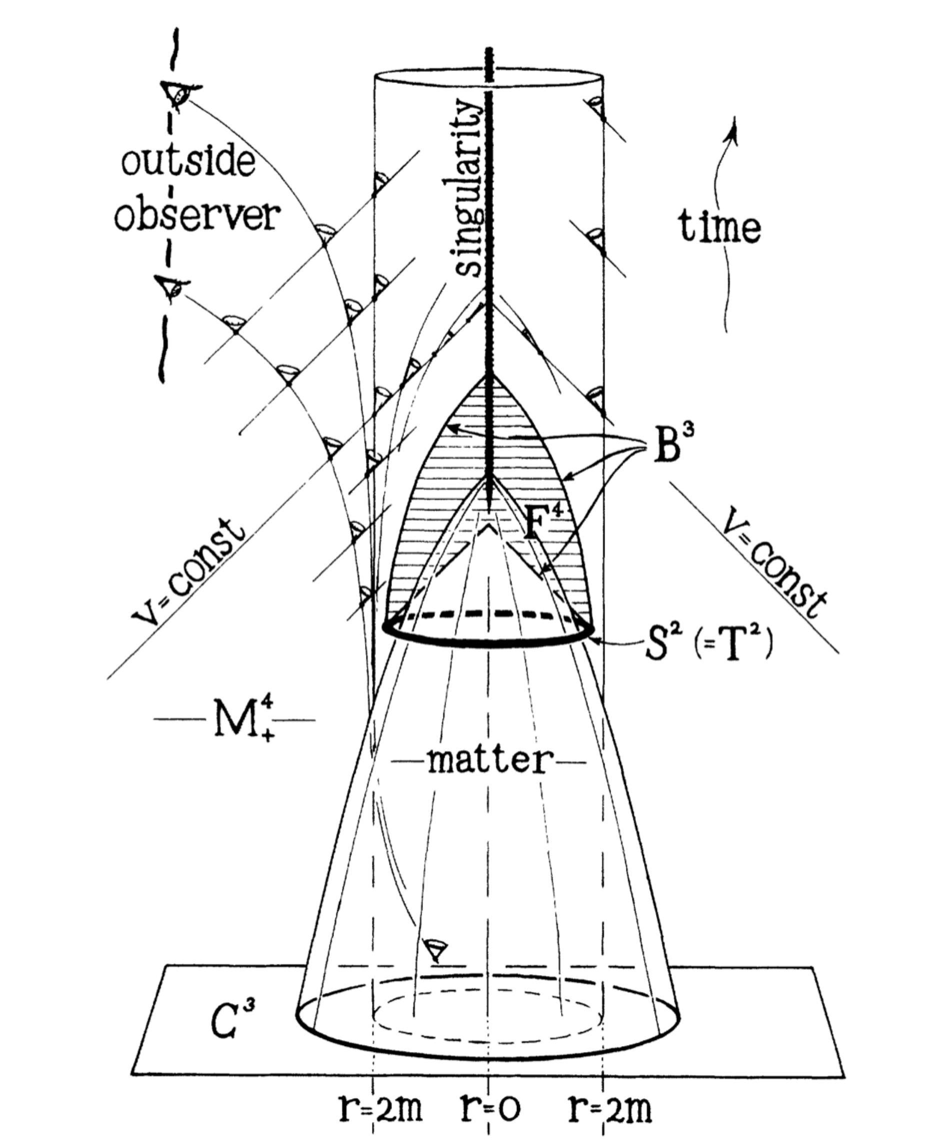
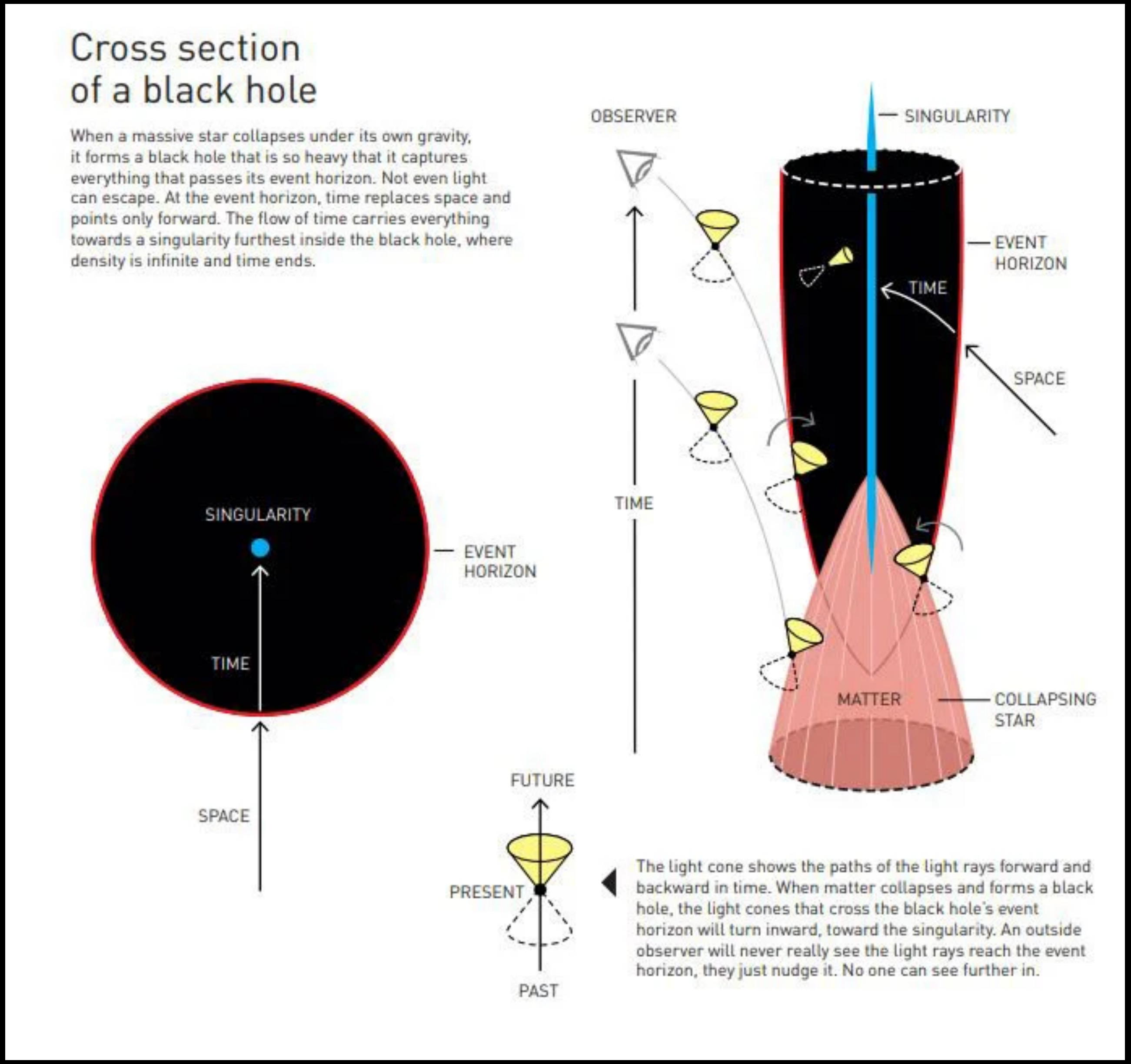


FIG. 1. Spherically symmetrical collapse (one space dimension surpressed). The diagram essentially also serves for the discussion of the asymmetrical case.

Penrose (1965)



https://www.nobelprize.org/prizes/physics/2020/popular-information/

Is a black hole stable?: Perturbation equation

PHYSICAL REVIEW VOLUME 108, NUMBER 4 NOVEMBER 15, 1957

Stability of a Schwarzschild Singularity

TULLIO REGGE, *Istituto di Fisica della Università di Torino, Torino, Italy*

AND

JOHN A. WHEELER, *Palmer Physical Laboratory, Princeton University, Princeton, New Jersey*

(Received July 15, 1957)

It is shown that a Schwarzschild singularity, spherically symmetrical and endowed with mass, will undergo small vibrations about the spherical form and will therefore remain stable if subjected to a small nonspherical perturbation.

Perturbation of a non-spinning BH

Regge and Wheeler (1957) Zerilli (1970)

$$d^2Q/dr^{*2}+k_{\mathrm{eff}}{}^2(r)Q=0.$$

Here we have made the abbreviations

$$dr^*=\exp(\tfrac{1}{2}\lambda-\tfrac{1}{2}\nu)dr,$$

and

$$k_{\mathrm{eff}}{}^2=k^2-L(L+1)e^\nu/r^2+6m^*e^\nu/r^3,$$

odd parity

$$\frac{d^2R_{LM}{}^{(e)}}{dr^{*2}}+[\omega^2-V_L{}^{(e)}(r)]R_{LM}{}^{(e)}=S_{LM}, \qquad (18)$$

$$V_L{}^{(e)}(r)=\left(1-\frac{2m}{r}\right)\times\frac{2\lambda^2(\lambda+1)r^3+6\lambda^2mr^2+18\lambda m^2r+18m^3}{r^3(\lambda r+3m)^2}. \qquad (19)$$

even parity

PERTURBATIONS OF A ROTATING BLACK HOLE. I. FUNDAMENTAL EQUATIONS FOR GRAVITATIONAL, ELECTROMAGNETIC, AND NEUTRINO-FIELD PERTURBATIONS*

SAUL A. TEUKOLSKY†
California Institute of Technology, Pasadena

Received 1973 April 12

ABSTRACT

This paper derives linear equations that describe dynamical gravitational, electromagnetic, and neutrino-field perturbations of a rotating black hole. The equations decouple into a single gravitational equation, a single electromagnetic equation, and a single neutrino equation. Each of these equations is completely separable into ordinary differential equations. The paper lays the mathematical groundwork for later papers in this series, which will deal with astrophysical applications: stability of the hole, tidal friction effects, superradiant scattering of electromagnetic waves, and gravitational-wave processes.

Subject headings: black holes — gravitation — neutrinos — relativity — rotation

Perturbation of a spinning BH

Teukolsky (1973)

$$\left[\frac{(r^2+a^2)^2}{\Delta}-a^2\sin^2\theta\right]\frac{\partial^2\psi}{\partial t^2}+\frac{4Mar}{\Delta}\frac{\partial^2\psi}{\partial t\partial\varphi}+\left[\frac{a^2}{\Delta}-\frac{1}{\sin^2\theta}\right]\frac{\partial^2\psi}{\partial\varphi^2}$$
$$-\Delta^{-s}\frac{\partial}{\partial r}\left(\Delta^{s+1}\frac{\partial\psi}{\partial r}\right)-\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right)-2s\left[\frac{a(r-M)}{\Delta}+\frac{i\cos\theta}{\sin^2\theta}\right]\frac{\partial\psi}{\partial\varphi}$$
$$-2s\left[\frac{M(r^2-a^2)}{\Delta}-r-ia\cos\theta\right]\frac{\partial\psi}{\partial t}+(s^2\cot^2\theta-s)\psi=4\pi\Sigma T. \qquad (4.7)$$

Is a black hole stable?: ringdown

Damping oscillation. Stable.

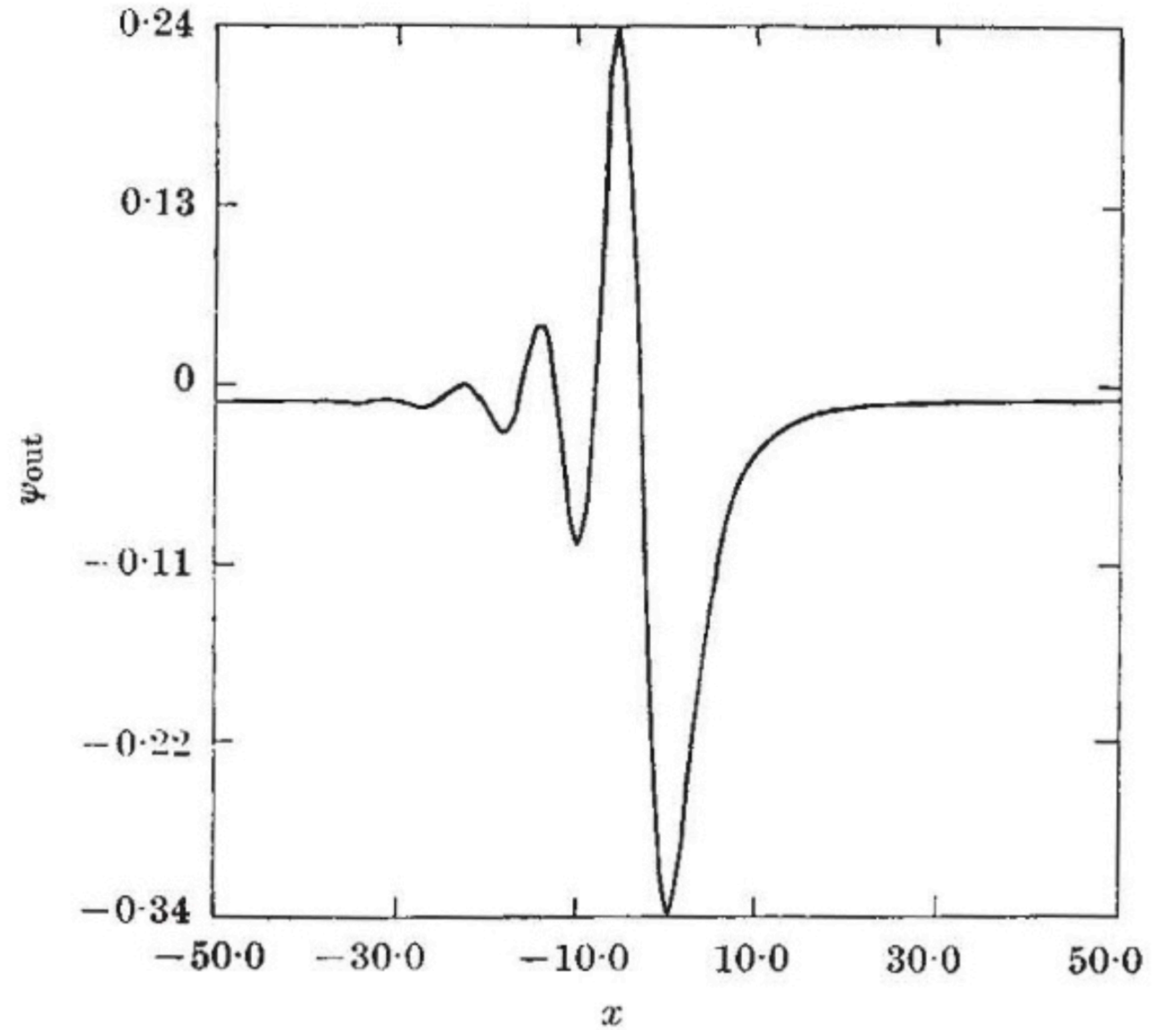


Fig. 3. The outgoing wave packet $\psi_{out}(x)$ at spatial infinity corresponding to the incident Gaussian wave packet $\psi_{in}(x) = e^{-ax^2}$ with $a = 1$.

Vishveshwara (1970)

Is a black hole stable?: quasinormal modes

Quasinormal mode (QNM) frequencies

Chandrasekhar and Detweiler (1975)

l	Zerilli's potential
2	$0.74734+0.17792i$
	$0.69687+0.54938i$
3	$1.19889+0.18541i$
	$1.16402+0.56231i$
	$0.85257+0.74546i$
4	$1.61835+0.18832i$
	$1.59313+0.56877i$
	$1.12019+0.84658i$

Therefore, a solution of Zerilli's equation with the asymptotic behaviour,

$$Z \to e^{+i\sigma r_*} + A\,e^{-i\sigma r_*} \quad (r_* \to +\infty)$$
$$\to \qquad B e^{+i\sigma r_*} \quad (r_* \to -\infty), \tag{62}$$

will yield a solution of equation (58) with the behaviour

$$X \to e^{+i\sigma r_*} + \frac{\frac{2}{3}n(n+1)+2mi\sigma}{\frac{2}{3}n(n+1)-2mi\sigma} A\,e^{-i\sigma r_*} \quad (r_* \to +\infty)$$
$$\to \qquad B e^{+i\sigma r_*} \quad (r_* \to -\infty). \tag{63}$$

The equality of the reflexion and transmission coefficients, that are determined by the two equations, is now manifest.† It is also clear that the complex frequencies belonging to the quasi-normal modes of the two parities must be the same; and the modes themselves must be related by equation (61).

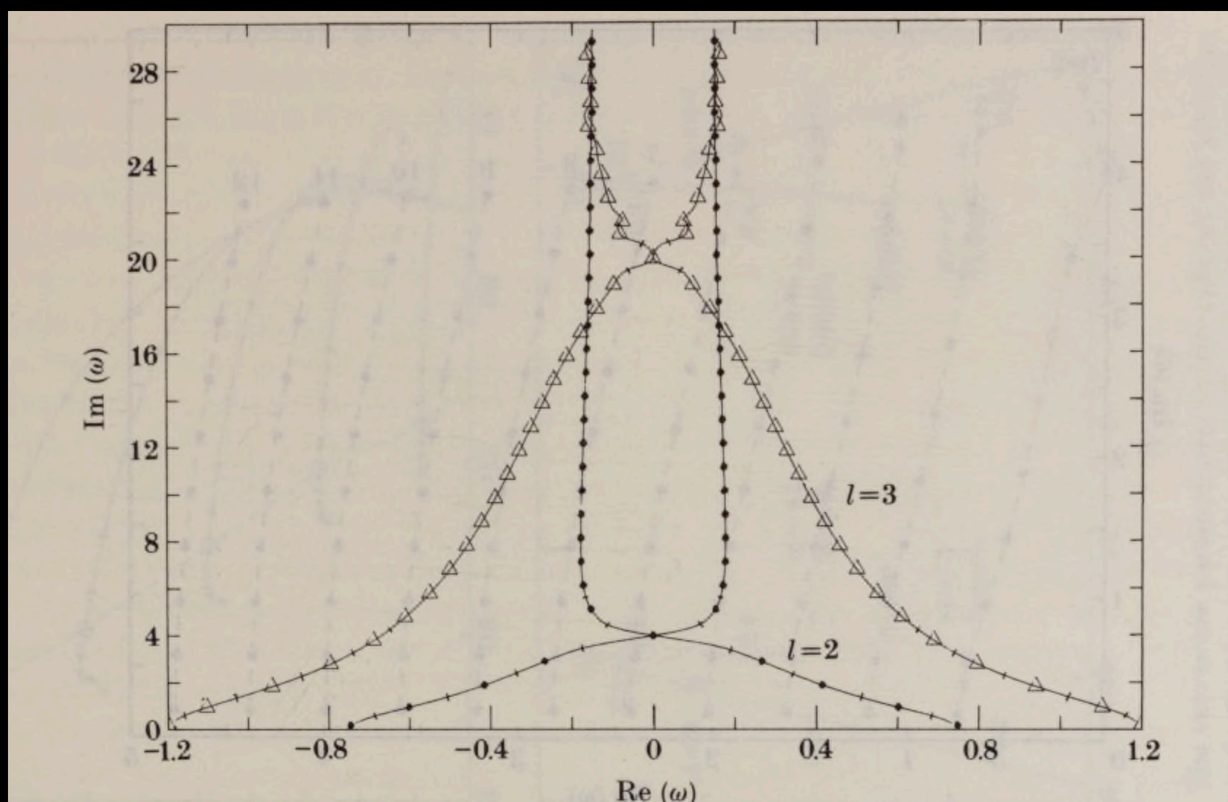
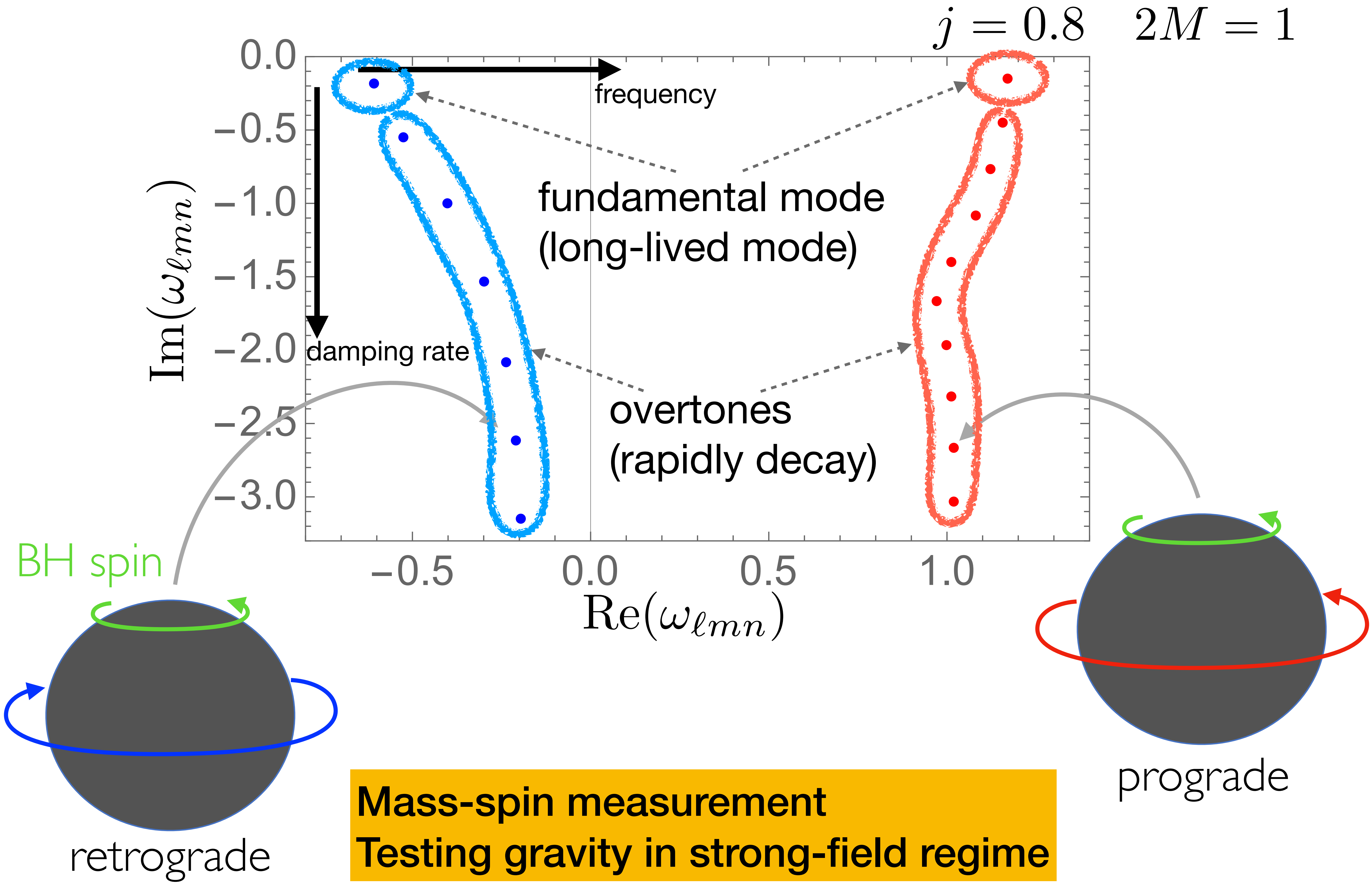


FIGURE 1. First 60 Schwarzschild quasi-normal frequencies for $l = 2$ and $l = 3$. The odd-order frequencies are prominently marked; a few-even order frequencies are indicated as short bars perpendicular to the curves connecting the points.

Leaver (1985)

QNMs and spin

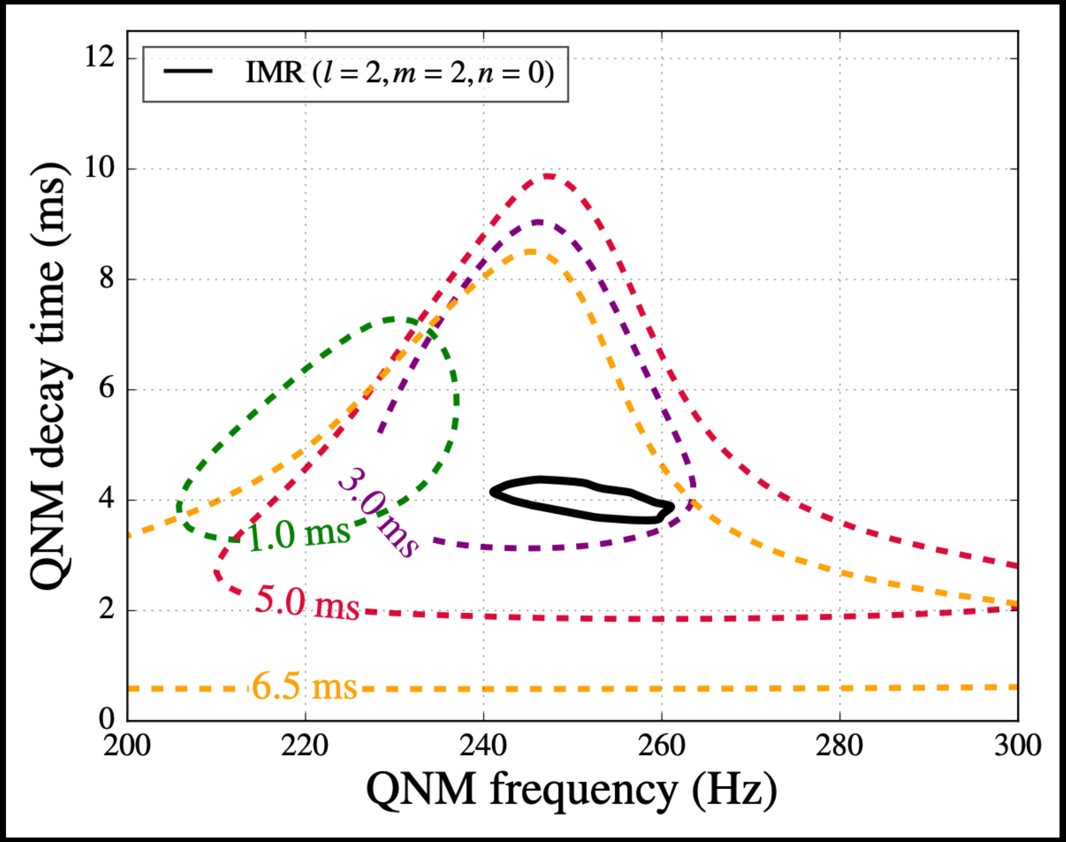


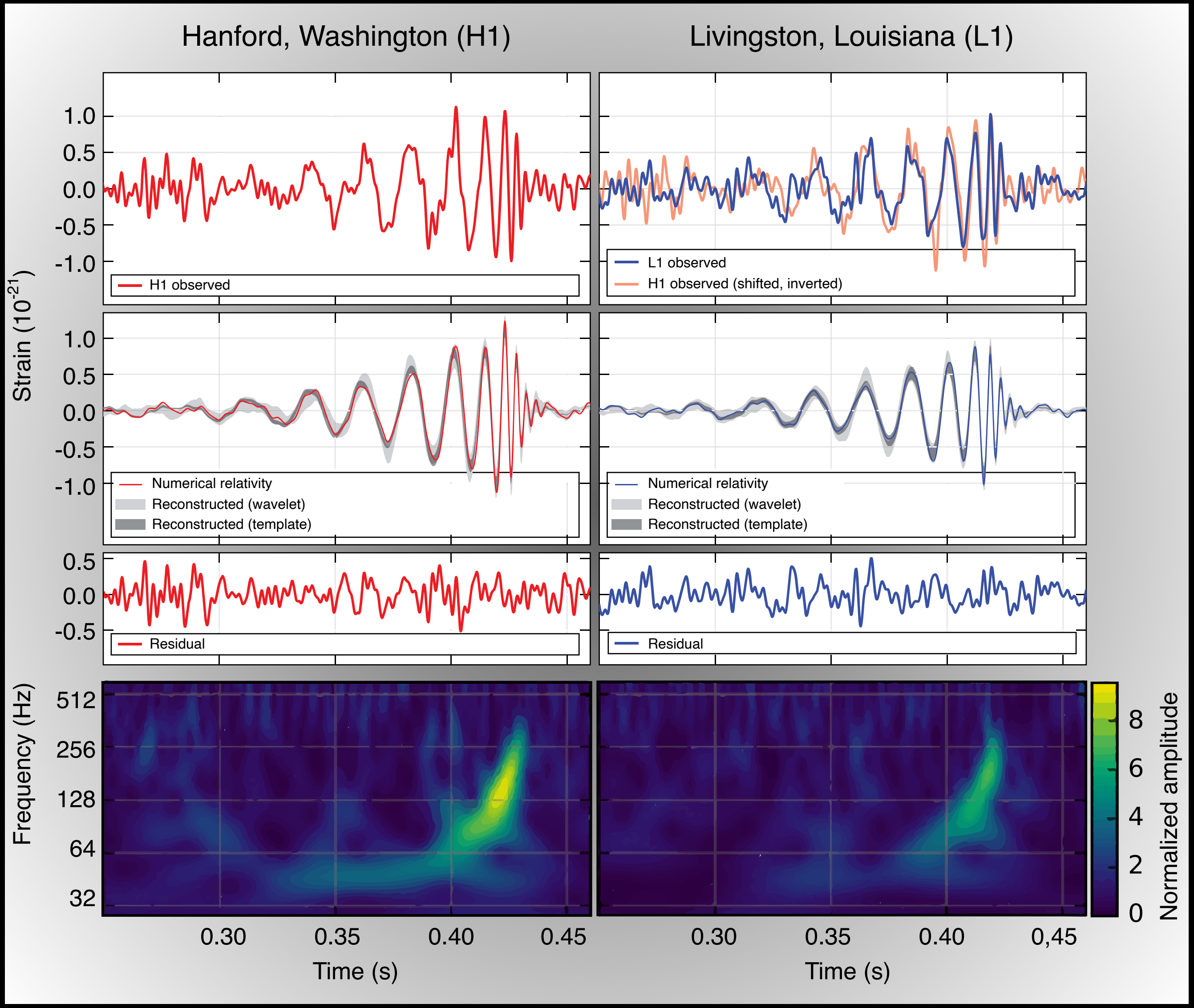
Mass-spin measurement
Testing gravity in strong-field regime

We heard the ringing!

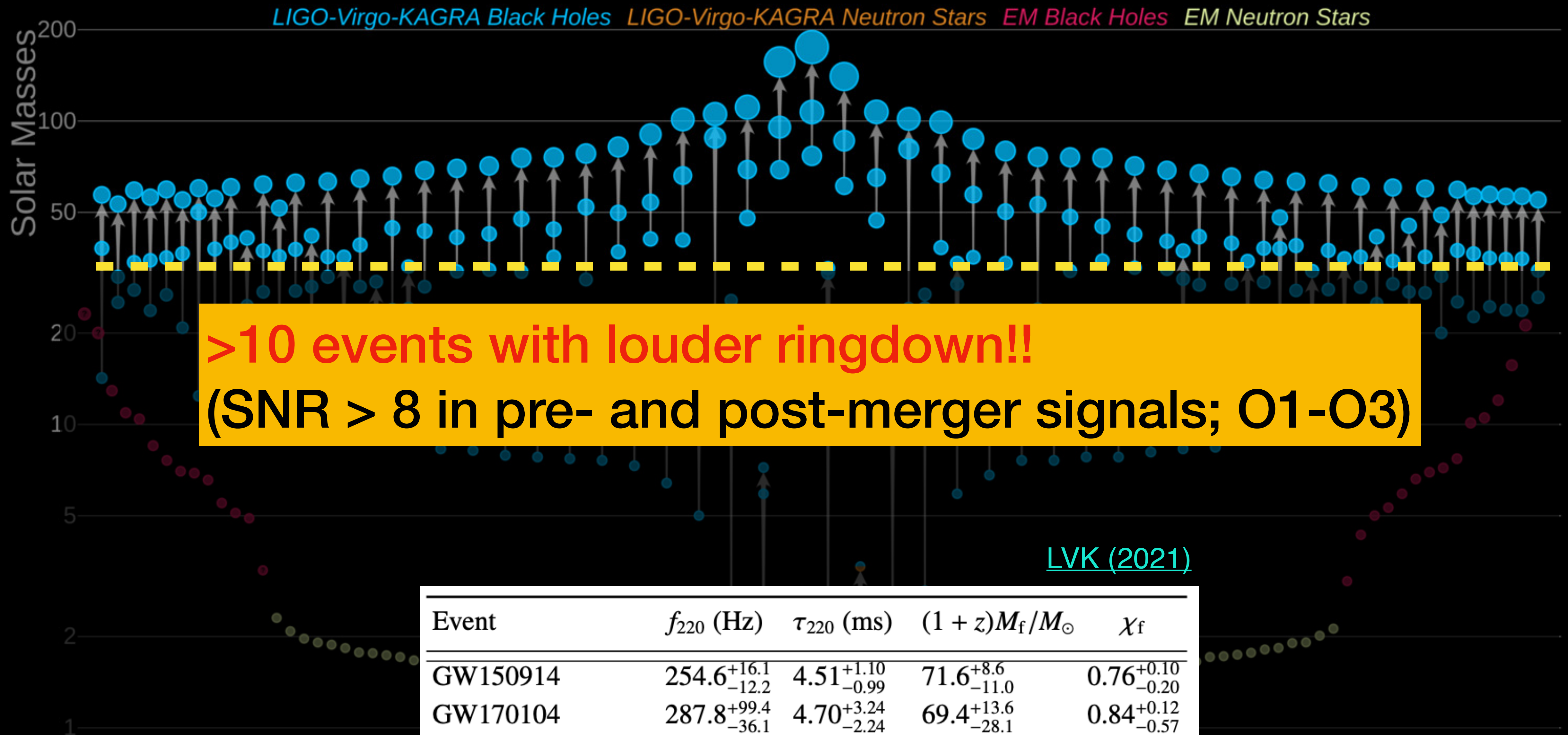
GW150914

LIGO (2016)





Masses in the Stellar Graveyard



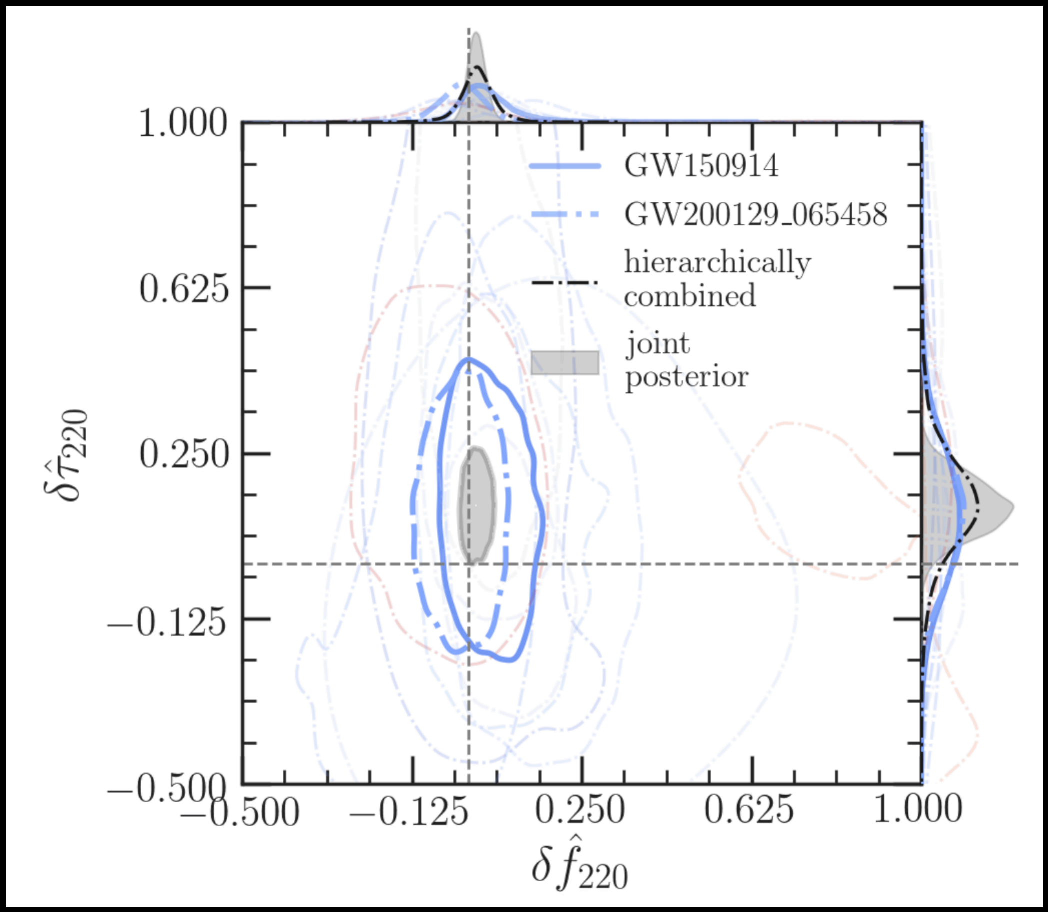
>10 events with louder ringdown!!
(SNR > 8 in pre- and post-merger signals; O1-O3)

LVK (2021)

Event	f_{220} (Hz)	τ_{220} (ms)	$(1+z)M_{\rm f}/M_{\odot}$	$\chi_{\rm f}$
GW150914	$254.6^{+16.1}_{-12.2}$	$4.51^{+1.10}_{-0.99}$	$71.6^{+8.6}_{-11.0}$	$0.76^{+0.10}_{-0.20}$
GW170104	$287.8^{+99.4}_{-36.1}$	$4.70^{+3.24}_{-2.24}$	$69.4^{+13.6}_{-28.1}$	$0.84^{+0.12}_{-0.57}$
GW190519_153544	$123.6^{+11.9}_{-13.0}$	$10.33^{+3.56}_{-3.07}$	$155.5^{+24.0}_{-29.9}$	$0.81^{+0.10}_{-0.28}$
GW190521_074359	$204.6^{+14.6}_{-11.7}$	$5.32^{+1.48}_{-1.21}$	$86.4^{+12.2}_{-14.3}$	$0.73^{+0.12}_{-0.26}$
GW190630_185205	$247.0^{+29.0}_{-49.8}$	$3.86^{+2.25}_{-1.73}$	$65.7^{+18.3}_{-39.2}$	$0.62^{+0.26}_{-0.62}$
GW190828_063405	$254.3^{+20.2}_{-17.7}$	$6.22^{+2.53}_{-2.34}$	$83.1^{+11.1}_{-18.2}$	$0.89^{+0.06}_{-0.25}$
GW190910 112807	$174.2^{+11.7}_{-7.5}$	$9.52^{+3.13}_{-2.68}$	$123.5^{+14.7}_{-18.1}$	$0.90^{+0.05}_{-0.11}$
GW191109_010717	$136.6^{+11.2}_{-18.3}$	$15.09^{+3.62}_{-2.74}$	$170.4^{+25.3}_{-15.1}$	$0.94^{+0.02}_{-0.04}$
GW200129_065458	$246.4^{+14.5}_{-18.1}$	$4.68^{+1.01}_{-0.97}$	$74.2^{+7.4}_{-10.0}$	$0.76^{+0.10}_{-0.22}$
GW200208_130117	$460.7^{+40.7}_{-271.7}$	$18.25^{+47.49}_{-14.10}$	$71.5^{+23.8}_{-11.1}$	$1.00^{+0.00}_{-0.45}$
GW200224_222234	$196.1^{+10.2}_{-9.6}$	$7.00^{+1.86}_{-1.71}$	$101.6^{+10.4}_{-14.0}$	$0.85^{+0.07}_{-0.16}$
GW200311_115853	$241.8^{+19.9}_{-20.0}$	$4.72^{+1.75}_{-1.45}$	$75.3^{+12.4}_{-17.4}$	$0.76^{+0.13}_{-0.39}$

Testing gravity / Exploring new physics

GWTC-3 LVK (2021)



In the SEOBNRv4HM model, starting from estimates of the initial binary's masses and spins, NR fits [156, 262] are used to predict the mass and spin of the remnant object, which are then used to predict the ringdown frequencies and damping times [64, 263]. Thus, the frequency and damping time of the $(\ell, \pm m, 0)$ QNM, $(f_{\ell m0}, \tau_{\ell m0})$ are functions of the initial masses and spins:

$$f_{\ell m0}^{\mathrm{GR}} = f_{\ell m0}^{\mathrm{GR}}(m_1, m_2, \chi_1, \chi_2)\,, \quad (14)$$

$$\tau_{\ell m0}^{\mathrm{GR}} = \tau_{\ell m0}^{\mathrm{GR}}(m_1, m_2, \chi_1, \chi_2)\,. \quad (15)$$

$$f_{\ell m0} = f_{\ell m0}^{\mathrm{GR}}\,(1 + \delta\hat{f}_{\ell m0})\,,$$

$$\tau_{\ell m0} = \tau_{\ell m0}^{\mathrm{GR}}\,(1 + \delta\hat{\tau}_{\ell m0})\,.$$

New physics or statistical error?

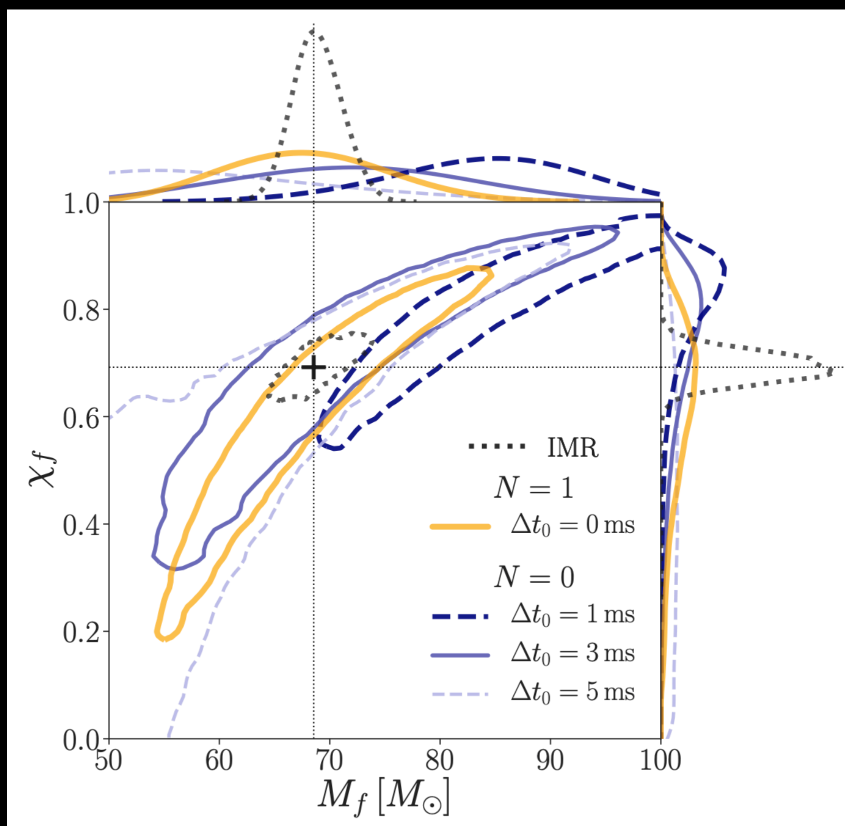
- Combining 12 event only.

- Bias from the prior ($\delta\hat{\tau}_{\ell m0} > -1$) non-negligible.

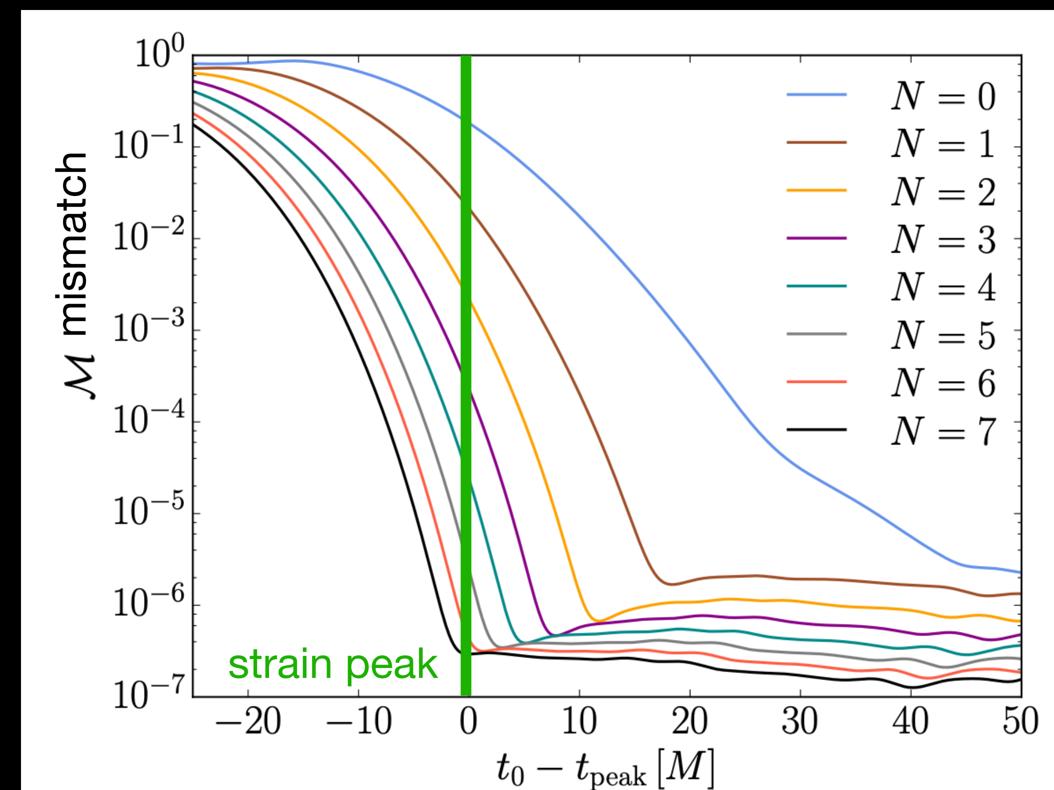
LVK's O4 and O5 will update!

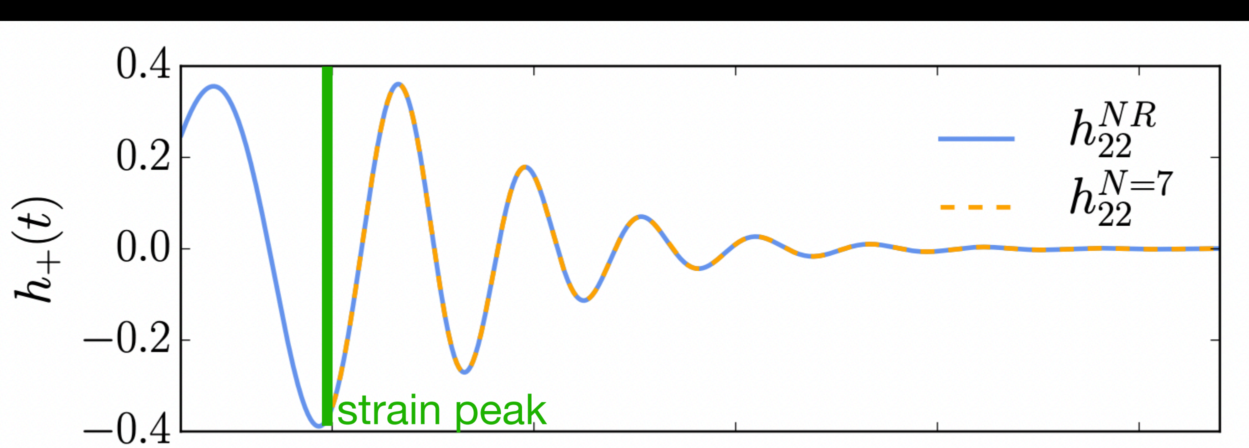
Overtones? Onset of QNM excitation?

Isi+ (2019)



Giesler+ (2019)





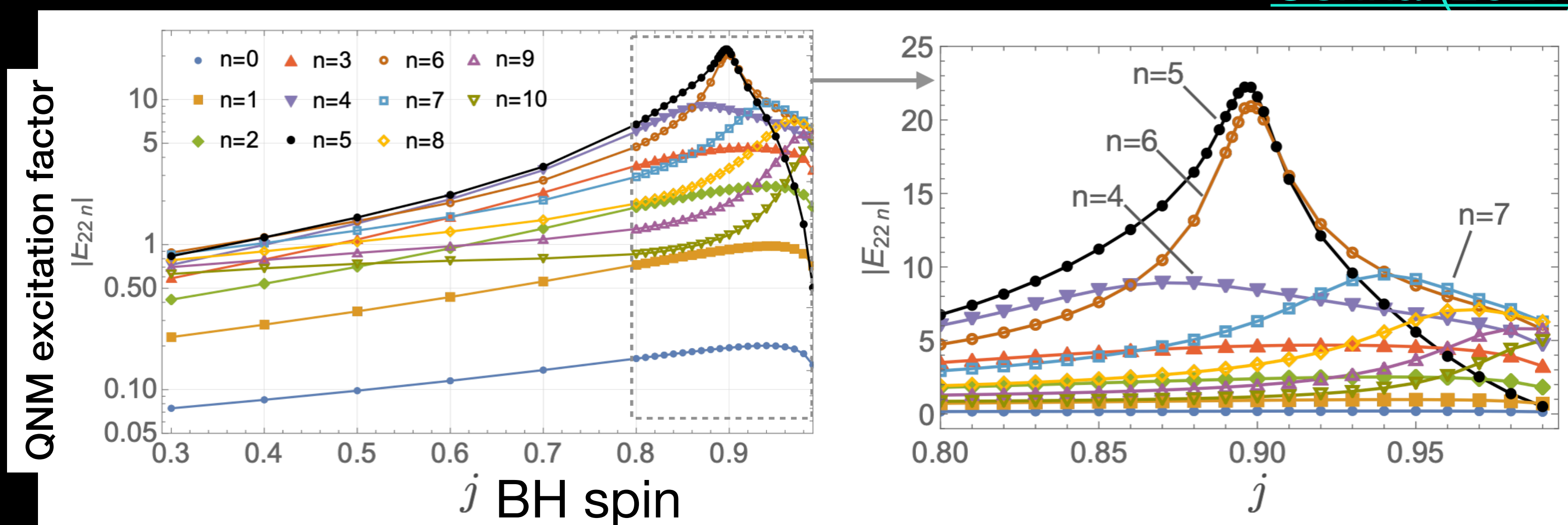
Fundamental mode ⟷ Higher overtones

N	A_0	A_1	A_2	A_3	A_4	A_5	A_6	A_7	$t_{\rm fit} - t_{\rm peak}$
0	0.971	-	-	-	-	-	-	-	47.00
1	0.974	3.89	-	-	-	-	-	-	18.48
2	0.973	4.14	8.1	-	-	-	-	-	11.85
3	0.972	4.19	9.9	11.4	-	-	-	-	8.05
4	0.972	4.20	10.6	16.6	11.6	-	-	-	5.04
5	0.972	4.21	11.0	19.8	21.4	10.1	-	-	3.01
6	0.971	4.22	11.2	21.8	28	21	6.6	-	1.50
7	0.971	4.22	11.3	23.0	33	29	14	2.9	0.00

Universal excitation

Leaver (1986)   Zhang, Berti, Cardoso (2013)   Oshita (2021)

Oshita (2021)



QNM excitation factor

$$h = \sum_n (E_{\ell mn} \times T_{\ell mn}) e^{-i\omega_{\ell mn} t}$$

GR   source

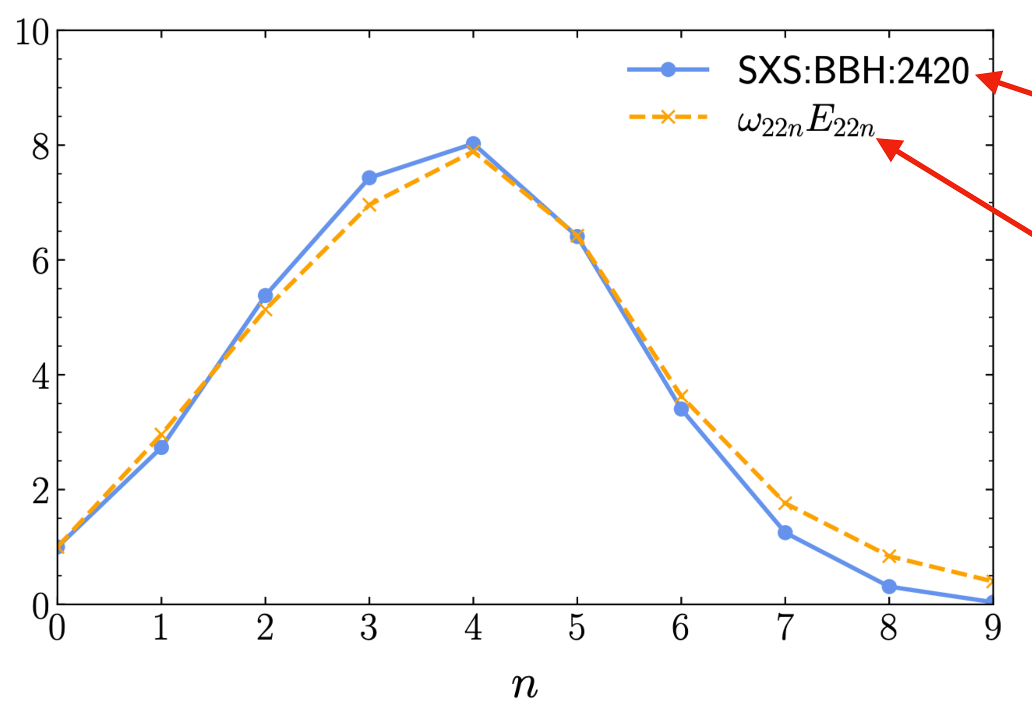


FIG. 12. The $(2, 2, n)$ NR amplitudes as measured in the moderate-spin case of SXS:BBH:2420 and the excitation factors $\omega_{22n}E_{22n}$ for $\chi = 0.75$. The excitation factors are computed using $t_0 = 2.5$ and both curves are normalized such that $C_{220} = \omega_{220}E_{220} = 1$.

QNM fit (NR + data analysis)

Theory (excitation factor)

Giesler, Ma, Mitman, Oshita, Teukolsky, et al. (2024)

Extraction of multiple QNMs? Controversial …

Recent studies claim that the inclusion of the fundamental mode and 7 overtones provides a very accurate description of the ringdown up to the peak strain amplitude, and significantly reduces the uncertainty in the extracted remnant properties [106]. However, we show that the higher overtones lead to very small mismatches by merely overfitting the waveforms. Furthermore, we argue that these higher overtones try to fit other physics (such as time variation in the QNM amplitudes due to initial data, an evolving spacetime background, and non-

Baibhav+ (2023)

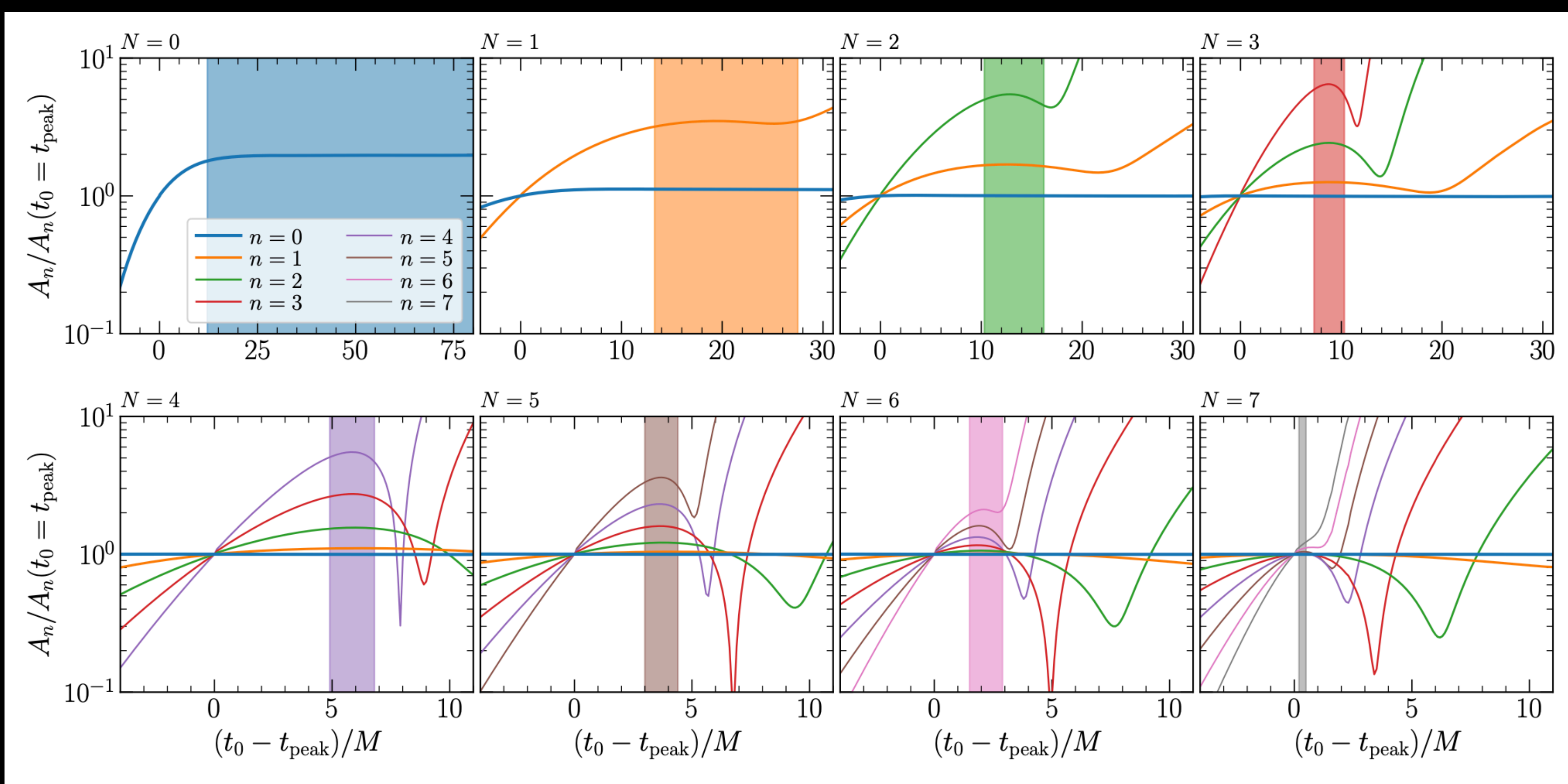
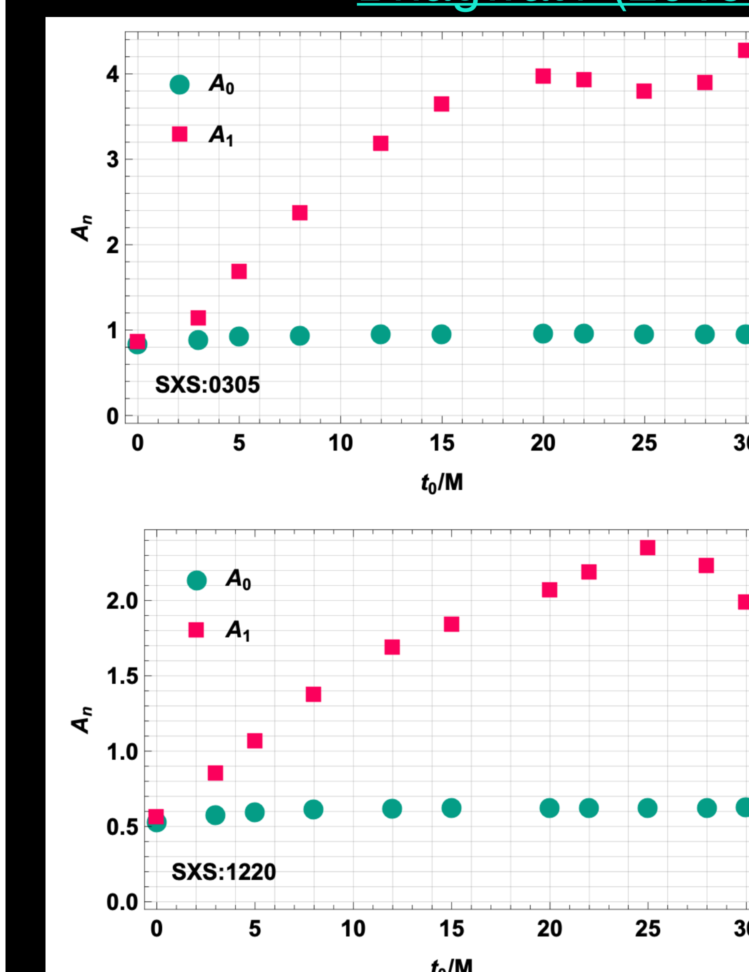


FIG. 6. Amplitude $A_n^N(t_0)/A_n^N(t_0 = t_{\rm peak})$ of QNMs as a function of the starting time t_0 for the SXS:BBH:0305 simulation. The shaded regions show the largest time range such that the amplitude of the highest overtone ($n = N$) is constant within 10%.

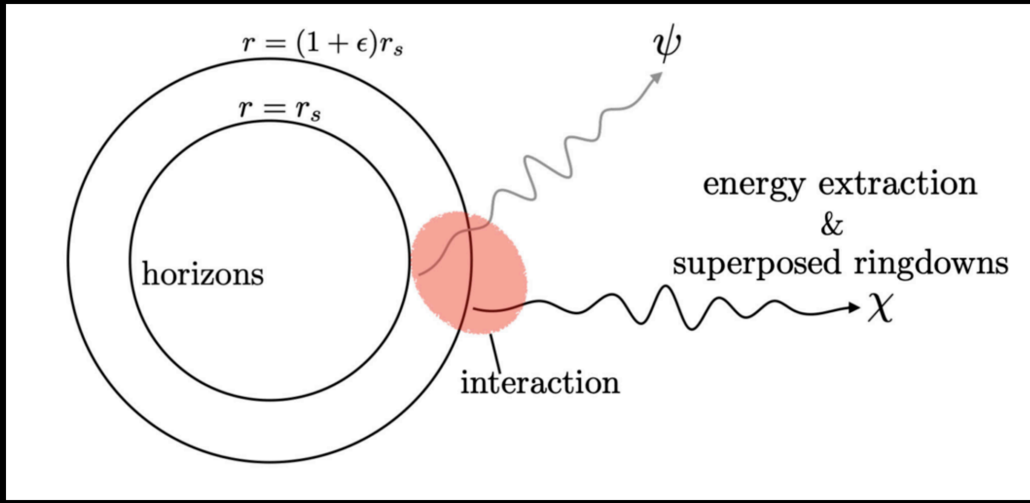
Bhagwat+ (2019)



Beyond general relativity

- Double-horizon black hole

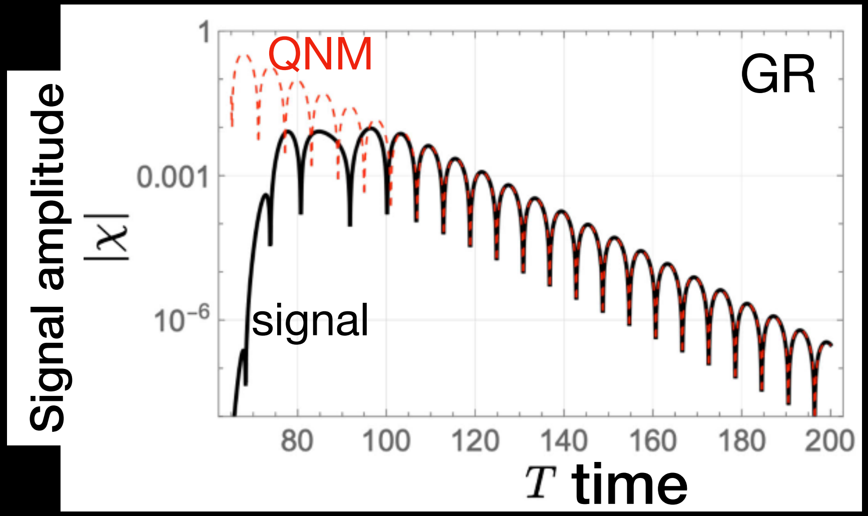
Cardoso, Mukohyama, NO, Takahashi (2024)

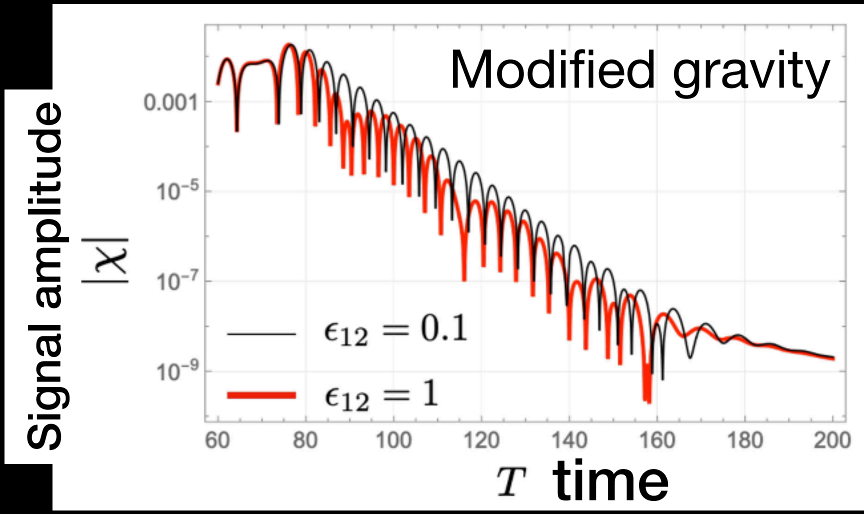


Ghost condensation N. Arkani-Hamed, H.C. Cheng, M. A. Luty, S. Mukohyama (2004)

Horava-Lifshitz theory P. Horava (2009)

khronometric theory D. Blas, O. Pujols, S. Sibiryakov (2011)





Simulation result with my numerical code

Anomalous modulation in ringdown signal

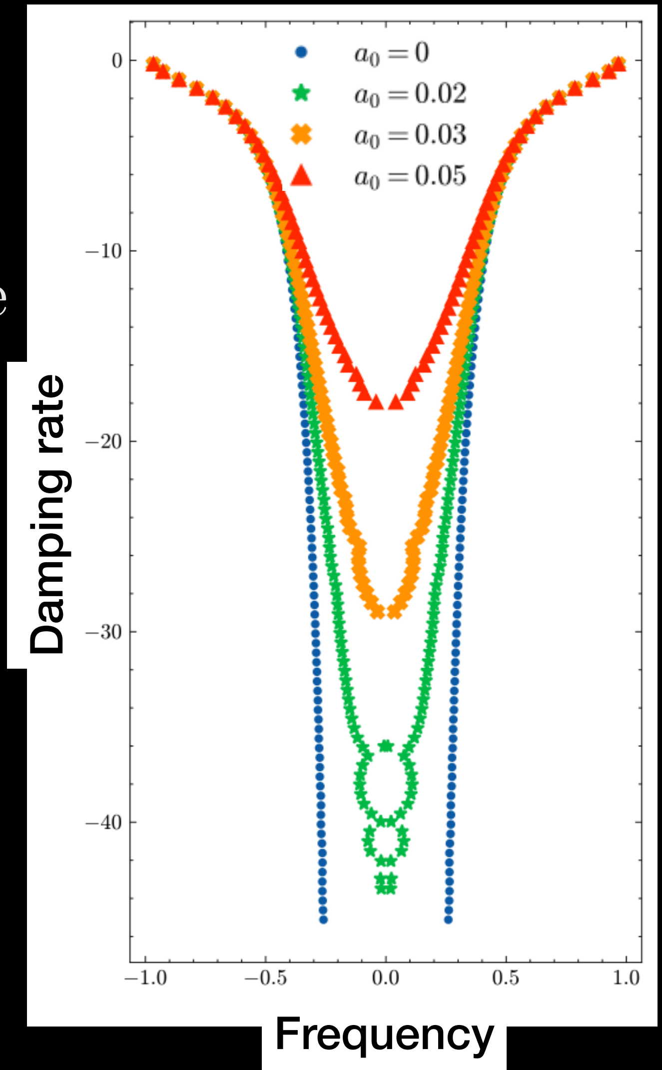
- QNM frequency of a BH in the loop quantum gravity

Livine, Montagnon, NO, Roussille (2024)

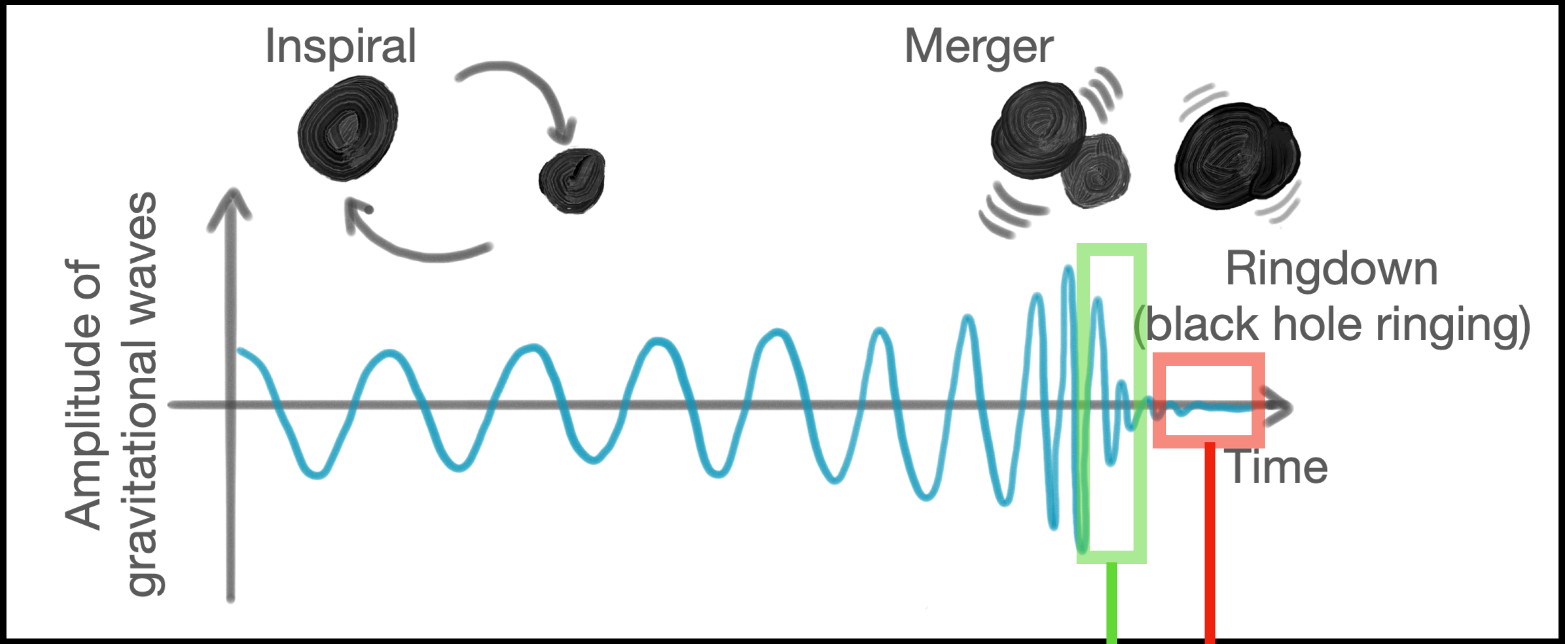
Motivated by loop quantum gravity

$a_0 \sim$ size of a BH core

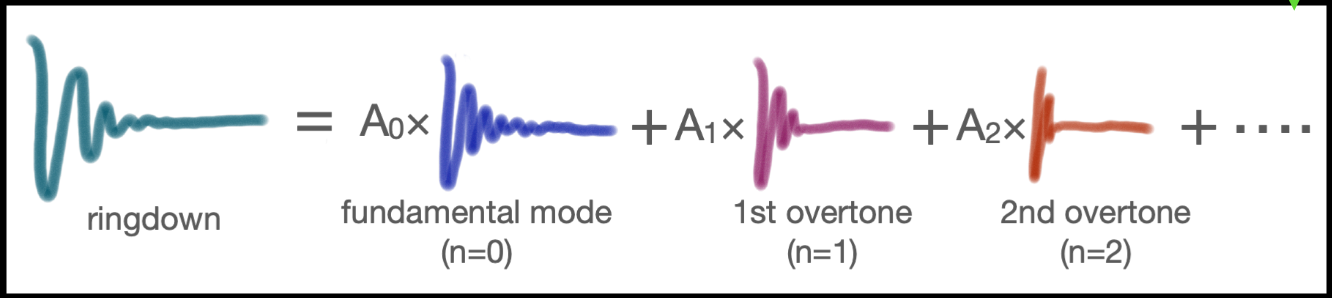
Drastic modification in highly-damped QNM frequencies



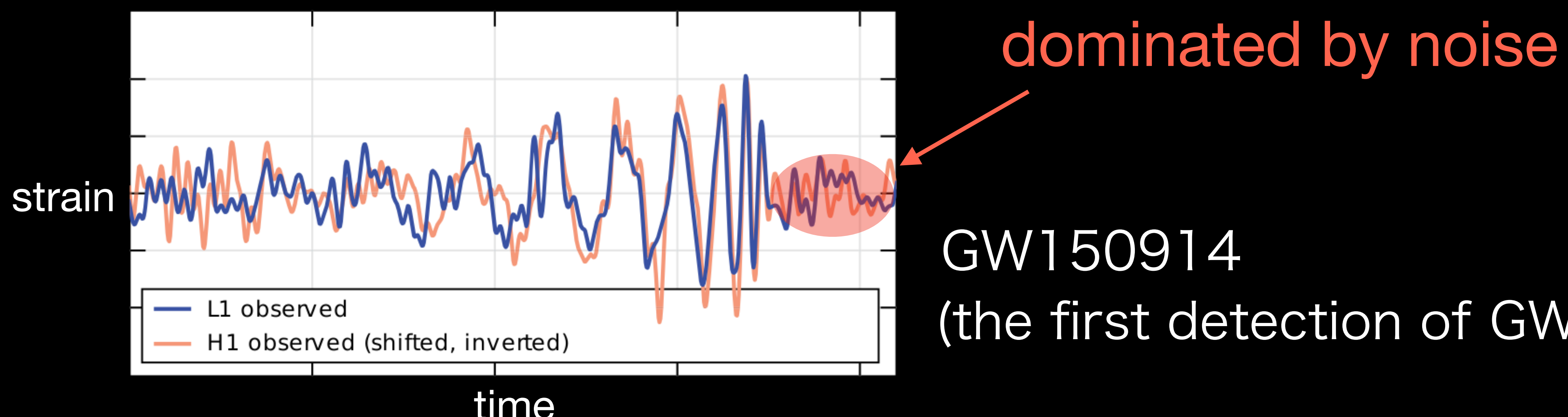
Is the remnant a BH?



Quasinormal modes

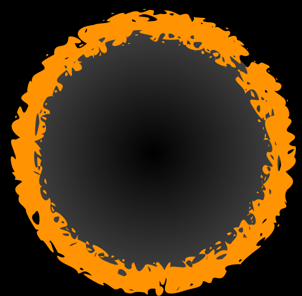


Near-horizon physics



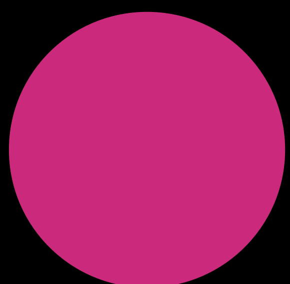
GW150914
(the first detection of GWs by LIGO)

Black hole “mimickers” or quantum-gravity models

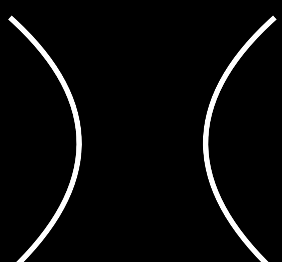


stretched horizon Susskind, et al. (1993)

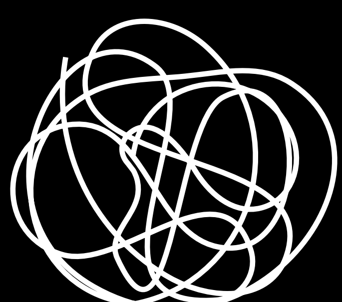
firewall Almheiri, et al. (2014)



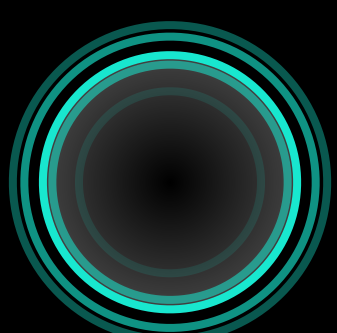
gravastar Mazur, et al. (2002)



wormhole Visser (1996)



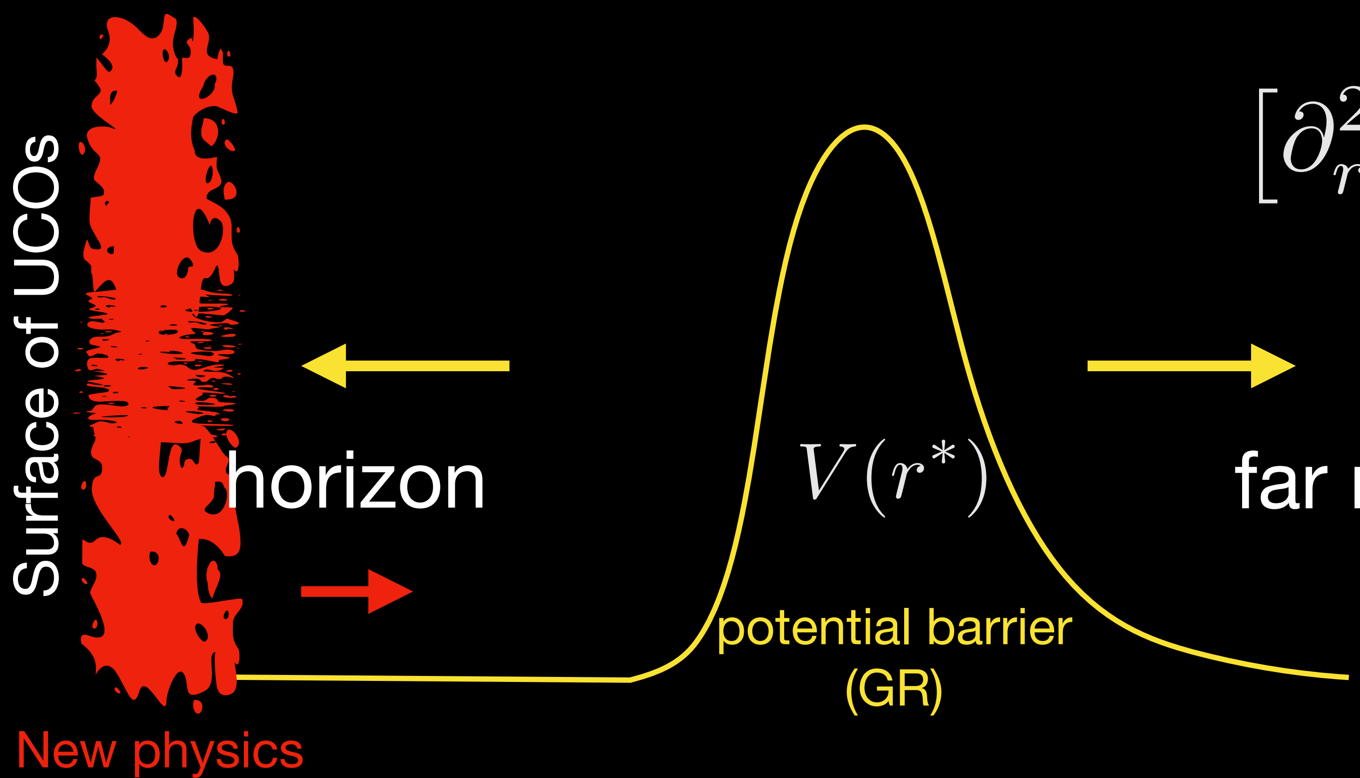
fuzzball Mathur, et al. (2002)



Black hole area quantization Bekenstein (1974)

Appear as black holes from a distance.
Differs from the standard picture near their surface.

GW echoes -phenomenological model-



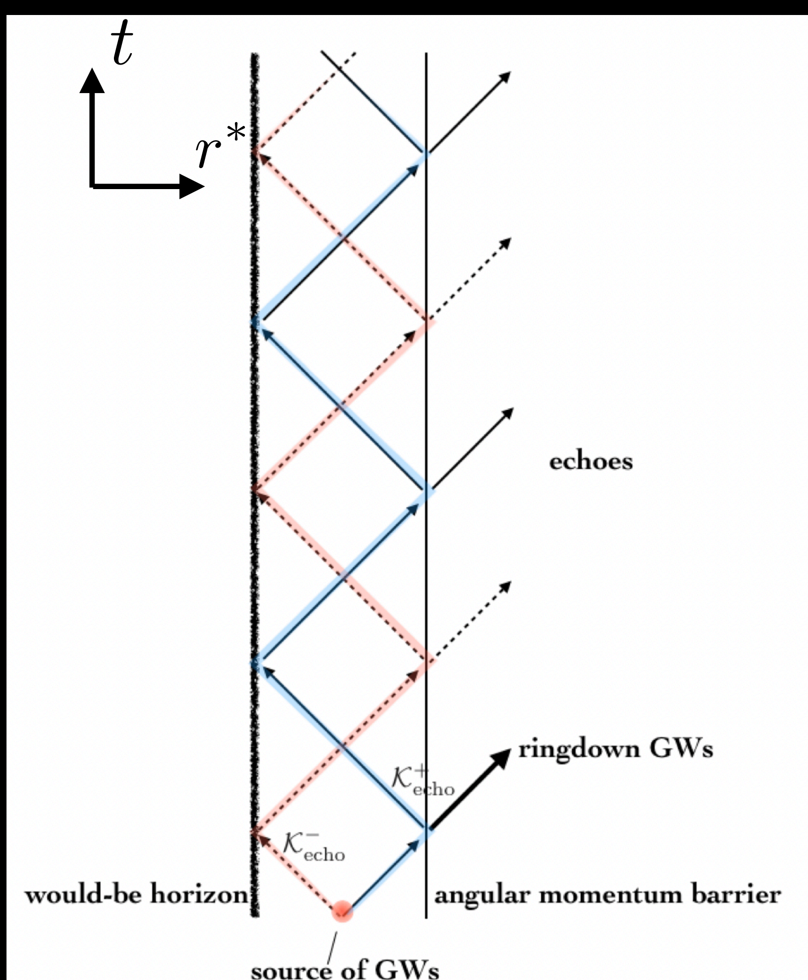
Perturbation equation

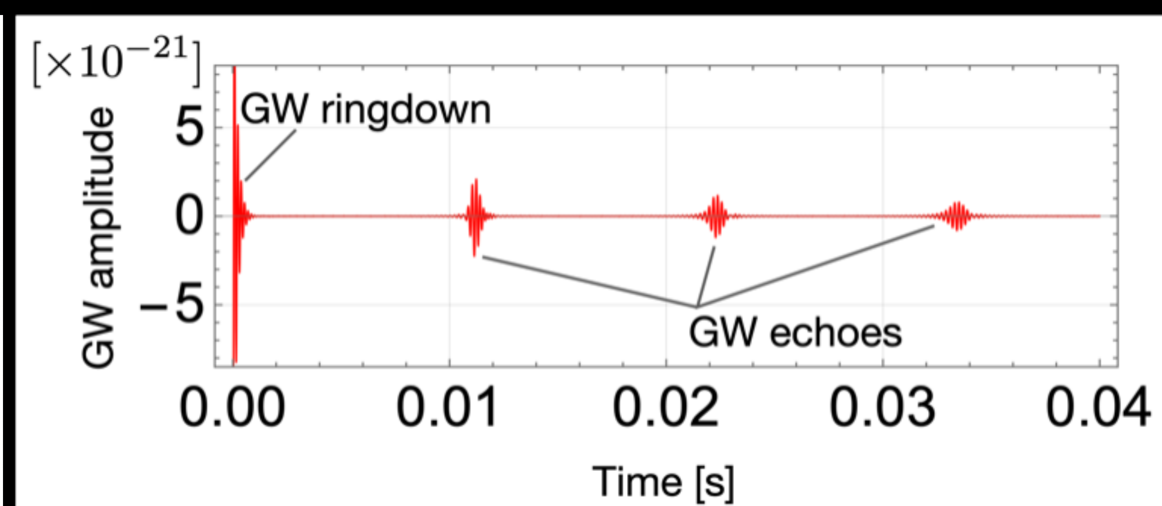
$$\left[\partial_{r^*}^2 + \omega^2 - V(r^*)\right] h_\omega = S_\omega(r^*)$$

potential barrier

perturbation variable

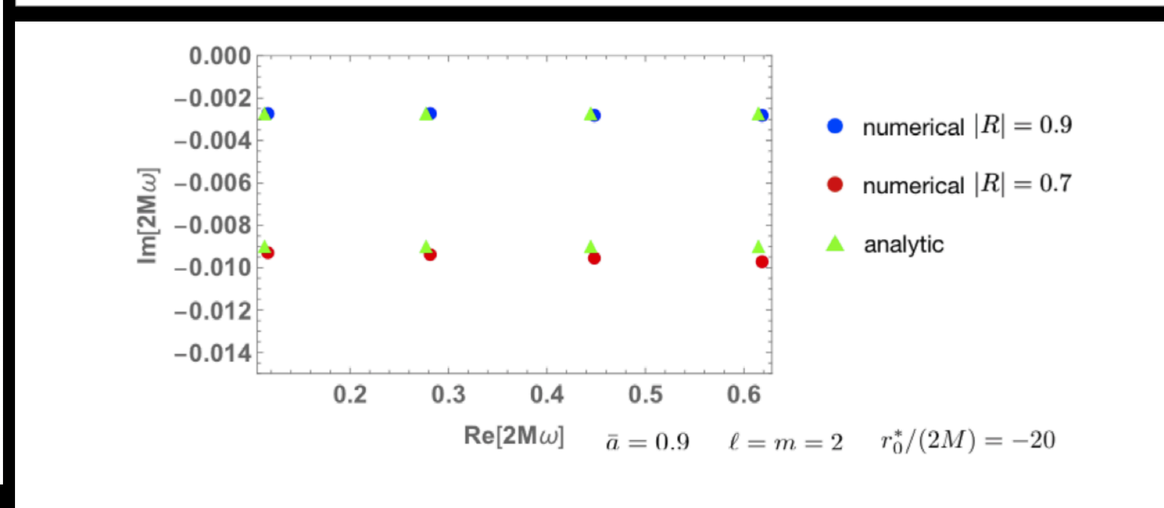
source





Time domain

NO, Tsuna, Afshordi (2020)



QNM

Abedi, Afshordi, NO, Wan (2020)

Probing quantum gravity and the third-generation gravitational-wave detectors

Abedi, Afshordi, NO, Wan (2020)

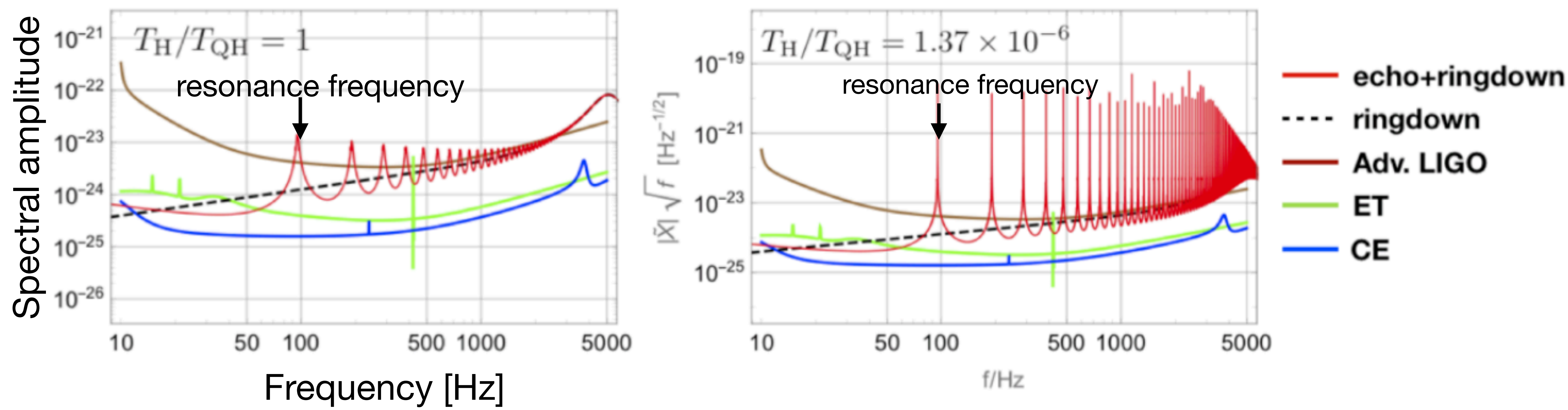
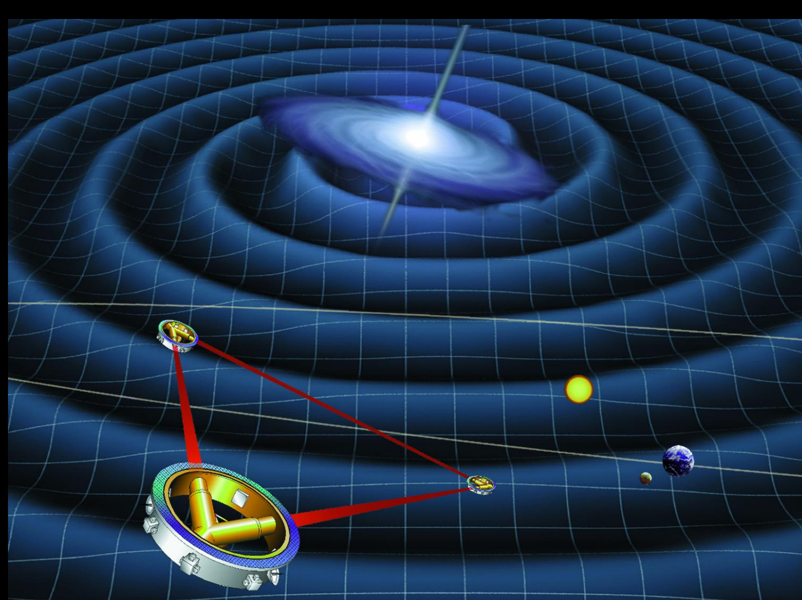


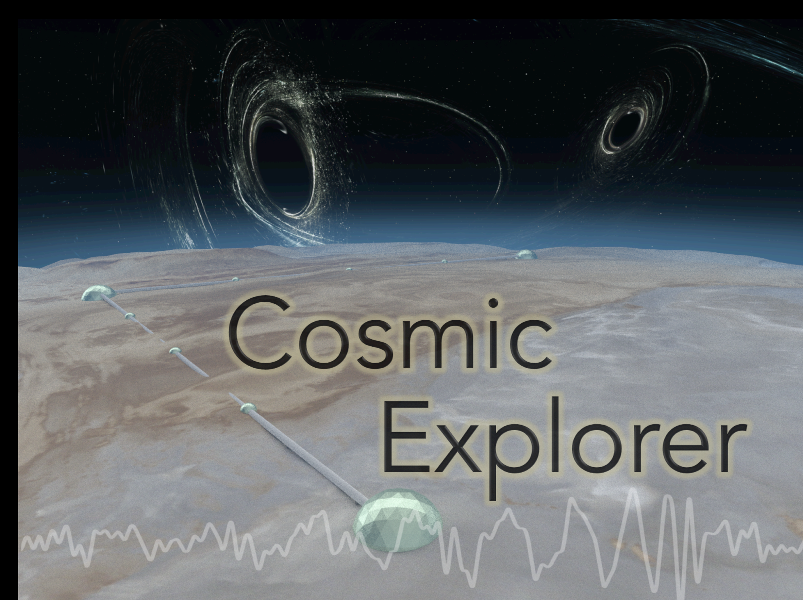
Figure 40. Spectra of ringdown and echo phases in the Boltzmann reflectivity model with $\bar{a} = 0.1$, $\epsilon_{rd} = 2.4 \times 10^{-6}$, $M = 2.4 M_\odot$, $\theta = 90°$, and $D_o = 1$ Mpc. Here we also assume $\gamma = 10^{-10}$, $T_H/T_{QH} = 1$ (left) and $T_H/T_{QH} = 1.37 \times 10^{-6}$ (right).

Laser Interferometer Space Antenna



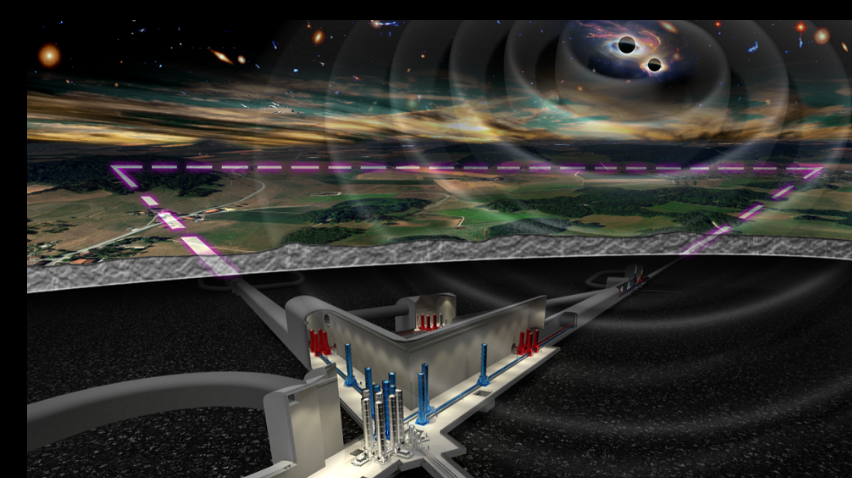
Mid-2030s NASA

Cosmic Explorer



1st 2030s 2nd 2040s

Einstein Telescope



2035

Computation of quasinormal modes

Direct integration or shooting method (numerical, lower tones)
Chandrasekhar and Detweiler (1975)

Leaver's method (numerical [Newton method], high accuracy, any overtones (?) in principle)
Leaver (1985)

WKB approximation (analytic, accurate for lower tones and for higher angular modes)
Schutz and Will (1985)

Monodromy technique (analytic, accurate for higher tones, difficult to look for a proper contour)
Motl and Neitzke (2003)

Spectral method (very new!)
Chung, Wagle, Yunes (2023)

Exact WKB technique (analytic, accurate for higher tones, straightforward to set a contour)
Miyachi, Namba, Omiya, Oshita, in prep.

…and more

Definition of QNM frequencies

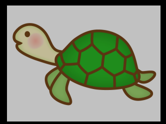
Black hole's "free oscillation" → excitation of BH QNMs

wave equation (Regge-Wheeler, Teukolsky, …)

$$\left[\partial_{r^*}^2 + \omega^2 - V(r^*)\right]\psi_\omega(r^*) = S_\omega(r^*)$$

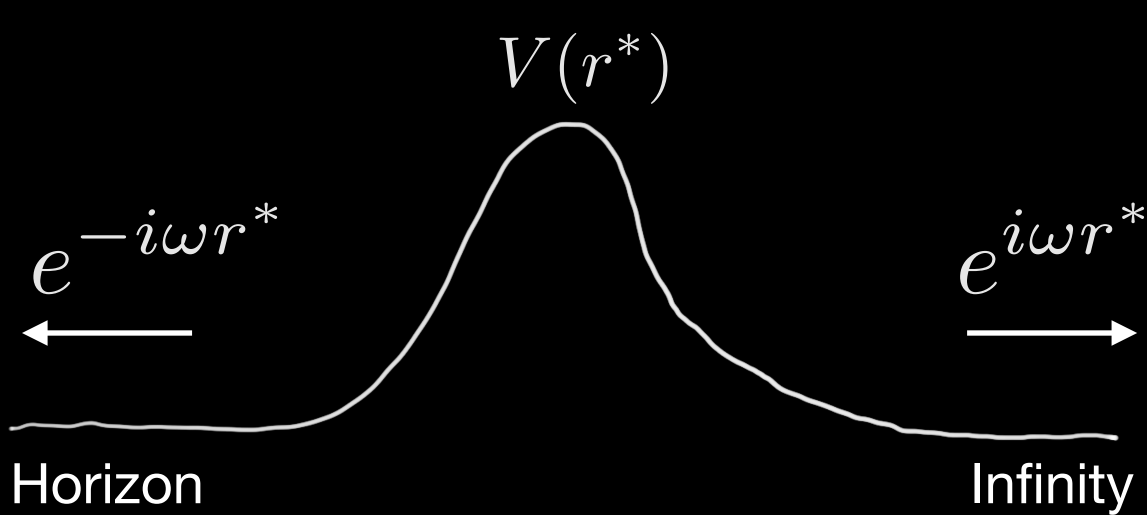
e.g., for Schwarzschild

$$r^* \equiv r + r_g \log(r/r_g - 1)$$

horizon $r \to r_+, \; r^* \to -\infty$ 

infinity $r \to +\infty, \; r^* \to +\infty$

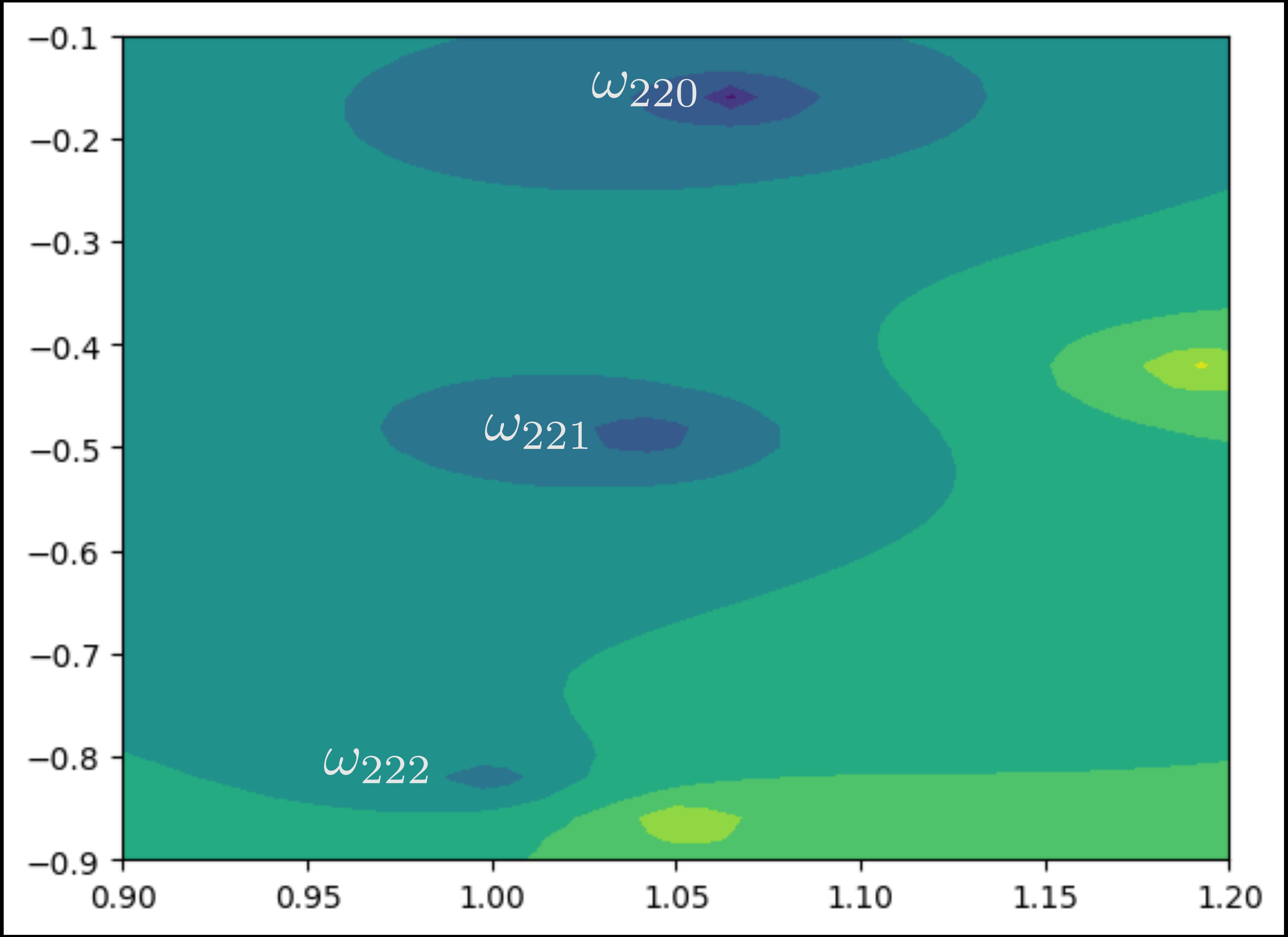
boundary condition



This B. C. is satisfied only at complex frequencies:

$$\omega = \omega_n \in \mathbb{C}$$

$2M = 1,\ \chi = J/M^2 = 0.7$ $(\ell, m) = (2, 2)$



$\mathrm{Im}(\omega)$

$\mathrm{Re}(\omega)$

$\omega = \omega_{\ell mn}$

https://colab.research.google.com/drive/1Xe2MCXvLn-rcjdnVTNCwFozuxQfjo41P?usp=sharing

Leaver's method Leaver (1985)

Schwarzschild BH

Regge-Wheeler equation

$$r(r-1)\psi_{l,rr}+\psi_{l,r}-\left[\frac{\rho^2 r^3}{r-1}+l(l+1)-\frac{\epsilon}{r}\right]\psi_l=0$$

$$2M=1,\ \rho=-2iM\omega,\ \epsilon=s^2-1$$

$$s=\begin{cases}\pm 2, & \text{graviton}\\ \pm 1, & \text{photon}\\ 0 & \text{scalar}\end{cases}$$

Boundary condition

$$\psi_l \xrightarrow{r\to 1} (r-1)^{\rho} \ \text{ and } \ \psi_l \xrightarrow{r\to\infty} r^{-\rho}e^{-\rho r}$$

Ansatz; expansion of the QNM eigenfunction

$$\psi_l=(r-1)^{\rho}\, r^{-2\rho}\, e^{-\rho(r-1)}\sum_{n=0}^{\infty} a_n\left(\frac{r-1}{r}\right)^n$$

a_n: expansion coefficient

Plunging the ansatz into the Regge-Wheeler equation, we get:

Three-term recurrence relation

$$\begin{aligned} \alpha_0 a_1 + \beta_0 a_0 &= 0, \\ \alpha_n a_{n+1} + \beta_n a_n + \gamma_n a_{n-1} &= 0, \quad n = 1, 2 \ldots \end{aligned}$$

$a_0 \neq 0$ (Determine the scale of the solution)

$$\begin{aligned} \alpha_n &= n^2 + (2\rho + 2)n + 2\rho + 1 \;, \\ \beta_n &= -[2n^2 + (8\rho + 2)n + 8\rho^2 + 4\rho + l(l+1) - \epsilon] \;, \\ \gamma_n &= n^2 + 4\rho n + 4\rho^2 - \epsilon - 1 \;. \end{aligned}$$

"$n=0$ boundary condition"

$$\frac{a_1}{a_0} = -\frac{\beta_0}{\alpha_0}$$

Scale of the solution.

"$n=\infty$ boundary condition"

$$\frac{a_1}{a_0} = \frac{-\gamma_1}{\beta_1 -}\,\frac{\alpha_1\gamma_2}{\beta_2 -}\,\frac{\alpha_2\gamma_3}{\beta_3 -}\ldots$$

In practical, we truncate the expansion.

Imposing its convergence.

$$0 = \beta_0 - \frac{\alpha_0\gamma_1}{\beta_1 -}\,\frac{\alpha_1\gamma_2}{\beta_2 -}\,\frac{\alpha_2\gamma_3}{\beta_3 -}\ldots$$

On the improvement of the truncation,

see, Leaver (1985) Nollert (1992)

Kerr BH

$$\psi(t,r,\theta,\phi) = \frac{1}{2\pi}\int e^{-i\omega t}\sum_{l=|s|}^{\infty}\sum_{m=-l}^{l} e^{im\phi}S_{lm}(u)R_{lm}(r)d\omega$$

Two functions are unknown: radial and angular part.

Teukolsky equation(s)

$$\left[(1-u^2)S_{lm,u}\right]_{,u} + \left[a^2\omega^2u^2 - 2a\omega su + s + A_{lm} - \frac{(m+su)^2}{1-u^2}\right]S_{lm} = 0$$

$$\Delta R_{lm,rr} + (s+1)(2r-1)R_{lm,r} + V(r)R_{lm} = 0$$

$$V(r) = \left\{ \begin{array}{c} \left[(r^2+a^2)^2\omega^2 - 2am\omega r + a^2m^2 + is\left(am(2r-1) - \omega(r^2-a^2)\right)\right]\Delta^{-1} \\ + \left[2is\omega r - a^2\omega^2 - A_{lm}\right] \end{array} \right\}$$

$u = \cos\theta$

$A_{\ell m}$: separation constant

$\Delta = r^2 - r + a^2$

$$\left[(1-u^2)S_{lm,u}\right]_{,u} + \left[a^2\omega^2u^2 - 2a\omega su + s + A_{lm} - \frac{(m+su)^2}{1-u^2}\right]S_{lm} = 0$$

BCs. : regular at the poles.

$$S_{lm}(u) = e^{a\omega u}(1+u)^{\frac{1}{2}|m-s|}(1-u)^{\frac{1}{2}|m+s|}\sum_{n=0}^{\infty} a_n(1+u)^n$$

$$\Delta R_{lm,rr} + (s+1)(2r-1)R_{lm,r} + V(r)R_{lm} = 0$$

BCs. : ingoing at horizon & outgoing at infinity.

$$R_{lm}(r) \xrightarrow{r\to r_+} (r-r_+)^{-s-i\sigma_+} \quad \text{and} \quad R_{lm}(r) \xrightarrow{r\to\infty} r^{-1-2s+i\omega}\, e^{i\omega r}$$

$$R_{lm} = e^{i\omega r}(r-r_-)^{-1-s+i\omega+i\sigma_+}\,(r-r_+)^{-s-i\sigma_+}\sum_{n=0}^{\infty} d_n\left(\frac{r-r_+}{r-r_-}\right)^n$$

Angular part

$$0 = \beta_0^\theta - \frac{\alpha_0^\theta \gamma_1^\theta}{\beta_1^\theta -} \frac{\alpha_1^\theta \gamma_2^\theta}{\beta_2^\theta -} \frac{\alpha_2^\theta \gamma_3^\theta}{\beta_3^\theta - \ldots}$$

$$\begin{aligned}
\alpha_n^\theta &= -2(n+1)(n+2k_1+1) \\
\beta_n^\theta &= \; n(n-1) + 2n(k_1+k_2+1-2a\omega) \\
&\quad - [2a\omega(2k_1+s+1) - (k_1+k_2)(k_1+k_2+1)] - [a^2\omega^2 + s(s+1) + A_{lm}] \\
\gamma_n^\theta &= 2a\omega(n+k_1+k_2+s)
\end{aligned}$$

Radial part

$$0 = \beta_0^r - \frac{\alpha_0^r \gamma_1^r}{\beta_1^r -} \frac{\alpha_1^r \gamma_2^r}{\beta_2^r -} \frac{\alpha_2^r \gamma_3^r}{\beta_3^r - \ldots}$$

$$\begin{aligned}
\alpha_n^r &= \; n^2 + (c_0+1)n + c_0, \\
\beta_n^r &= -2n^2 + (c_1+2)n + c_3, \\
\gamma_n^r &= \; n^2 + (c_2-3)n + c_4 - c_2 + 2,
\end{aligned}$$

$$\begin{aligned}
c_0 &= 1 - s - i\omega - \frac{2i}{b}\left(\frac{\omega}{2} - am\right), \\
c_1 &= -4 + 2(2+b)i\omega + \frac{4i}{b}\left(\frac{\omega}{2} - am\right), \\
c_2 &= s + 3 - 3i\omega - \frac{2i}{b}\left(\frac{\omega}{2} - am\right), \\
c_3 &= \omega^2(4 + 2b - a^2) - 2am\omega - s - 1 - A_{lm} + (2+b)i\omega + \frac{4\omega + 2i}{b}\left(\frac{\omega}{2} - am\right), \\
c_4 &= s + 1 - 2\omega^2 - (2s+3)i\omega - \frac{4\omega + 2i}{b}\left(\frac{\omega}{2} - am\right).
\end{aligned}$$

Inversion technique

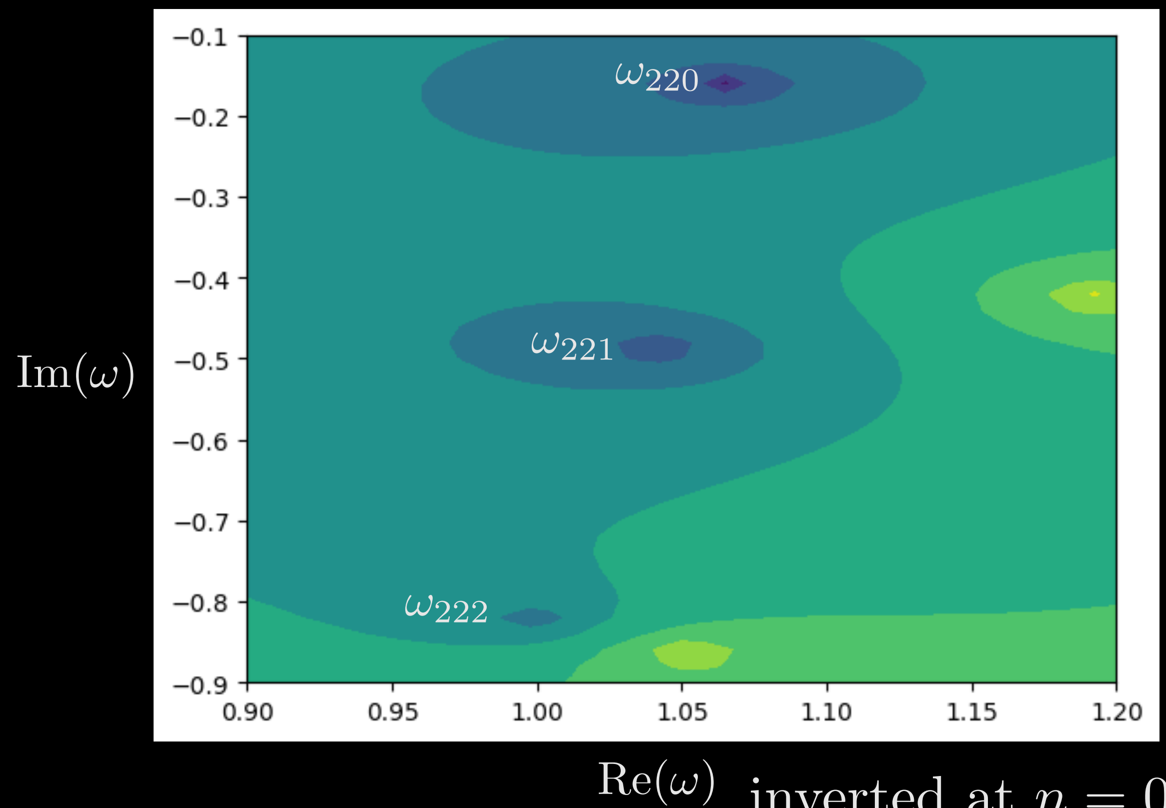
$$0 = \beta_0 - \frac{\alpha_0 \gamma_1}{\beta_1 -} \frac{\alpha_1 \gamma_2}{\beta_2 -} \frac{\alpha_2 \gamma_3}{\beta_3 -} \dots$$

is essentially equivalent to

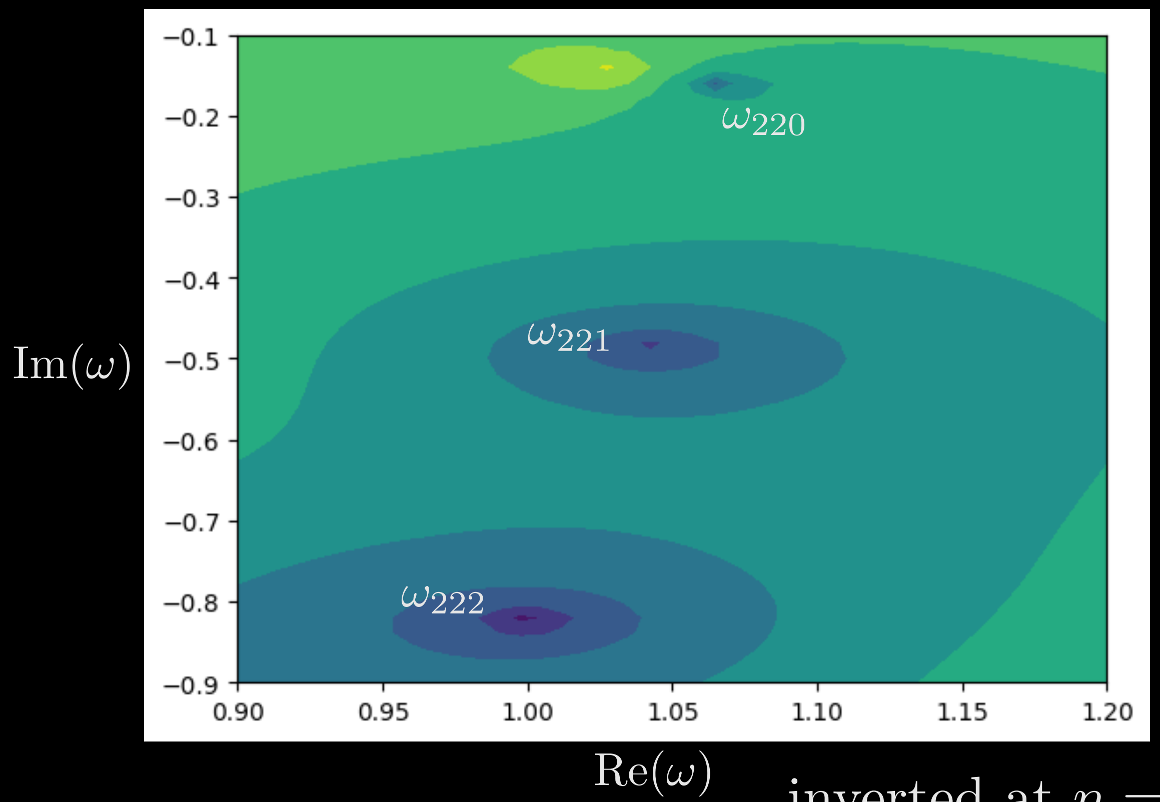
$$\left[\beta_n - \frac{\alpha_{n-1}\gamma_n}{\beta_{n-1} -} \frac{\alpha_{n-2}\gamma_{n-1}}{\beta_{n-2} -} \dots - \frac{\alpha_0\gamma_1}{\beta_0}\right] = \left[\frac{\alpha_n\gamma_{n+1}}{\beta_{n+1} -} \frac{\alpha_{n+1}\gamma_{n+2}}{\beta_{n+2} -} \frac{\alpha_{n+2}\gamma_{n+3}}{\beta_{n+3} -} \dots\right]$$

$$(n = 1, 2 \dots) \,.$$

The Newton method works very well for the nth overtone.



inverted at $n = 0$



inverted at $n = 2$

Continue to tutorial.