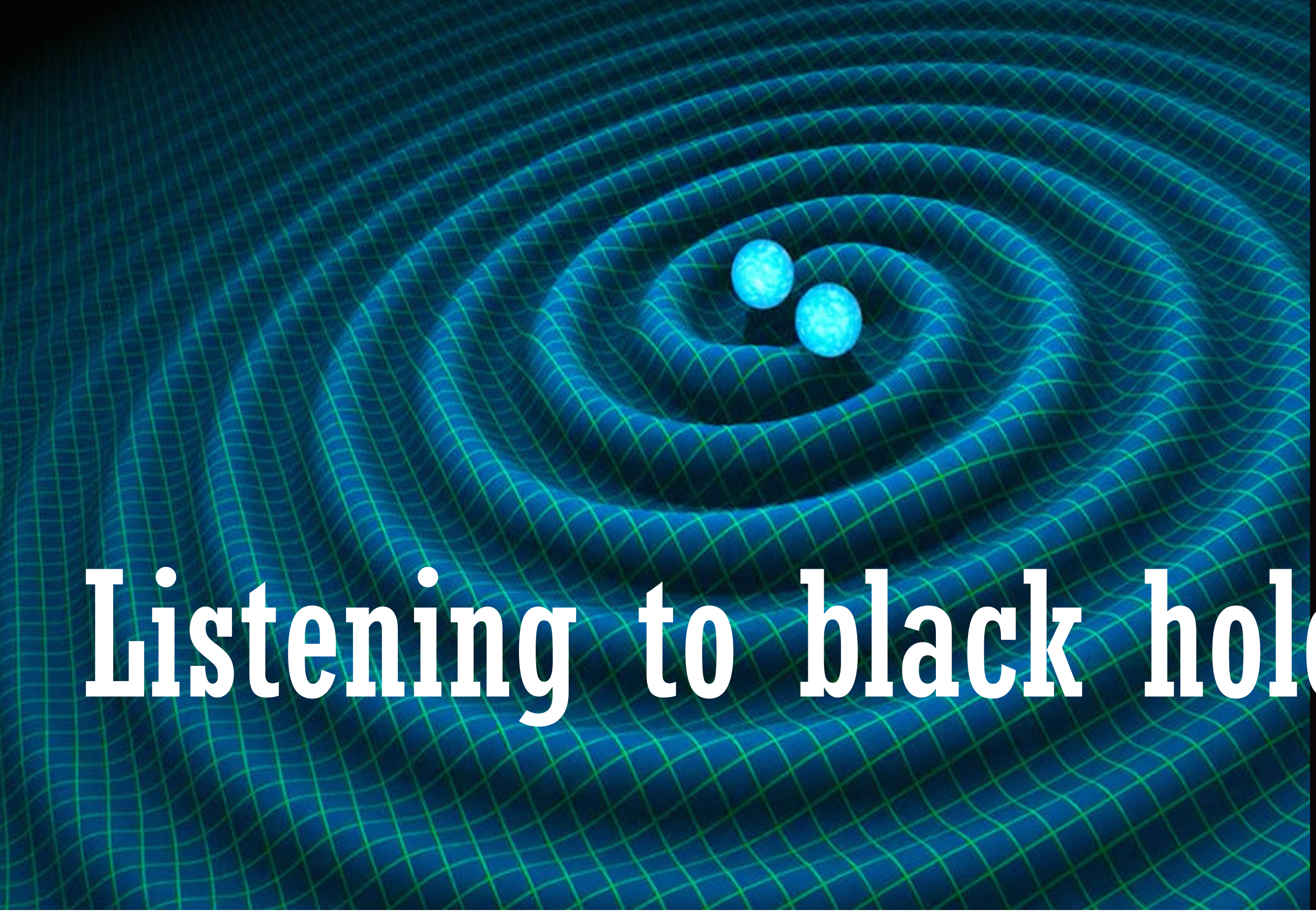

Listening to black hole portrayal

Che-Yu Chen

iTHEMS (SPDR), RIKEN, JAPAN

RIKEN's Programs for Junior Scientists

iTHEMS RIKEN Interdisciplinary Theoretical and Mathematical Sciences Program

- Black hole QNMs & WKB method
- Eikonal correspondence between QNMs and shadows
  - Basis
  - Testing gravity
- Conclusions

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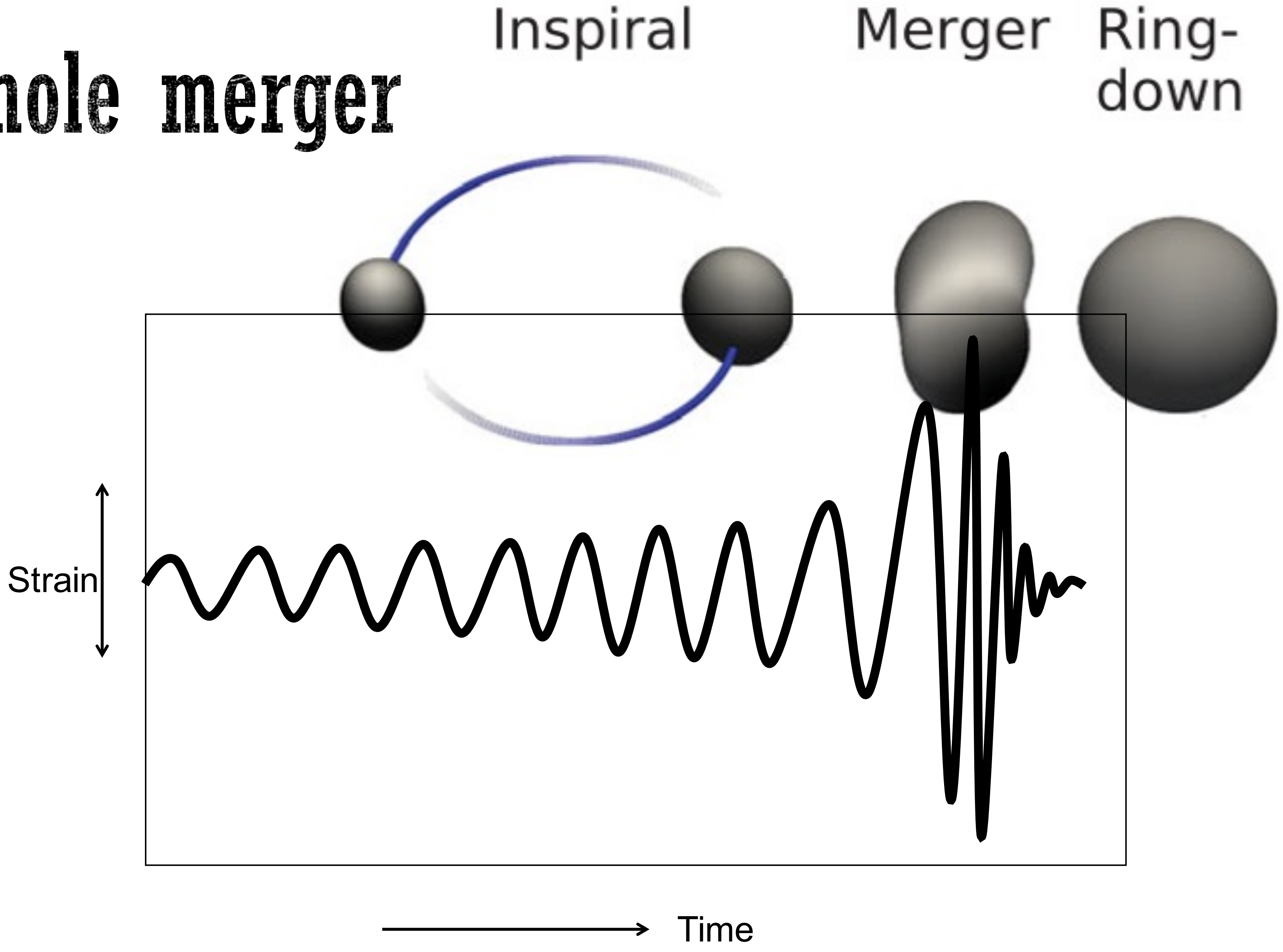
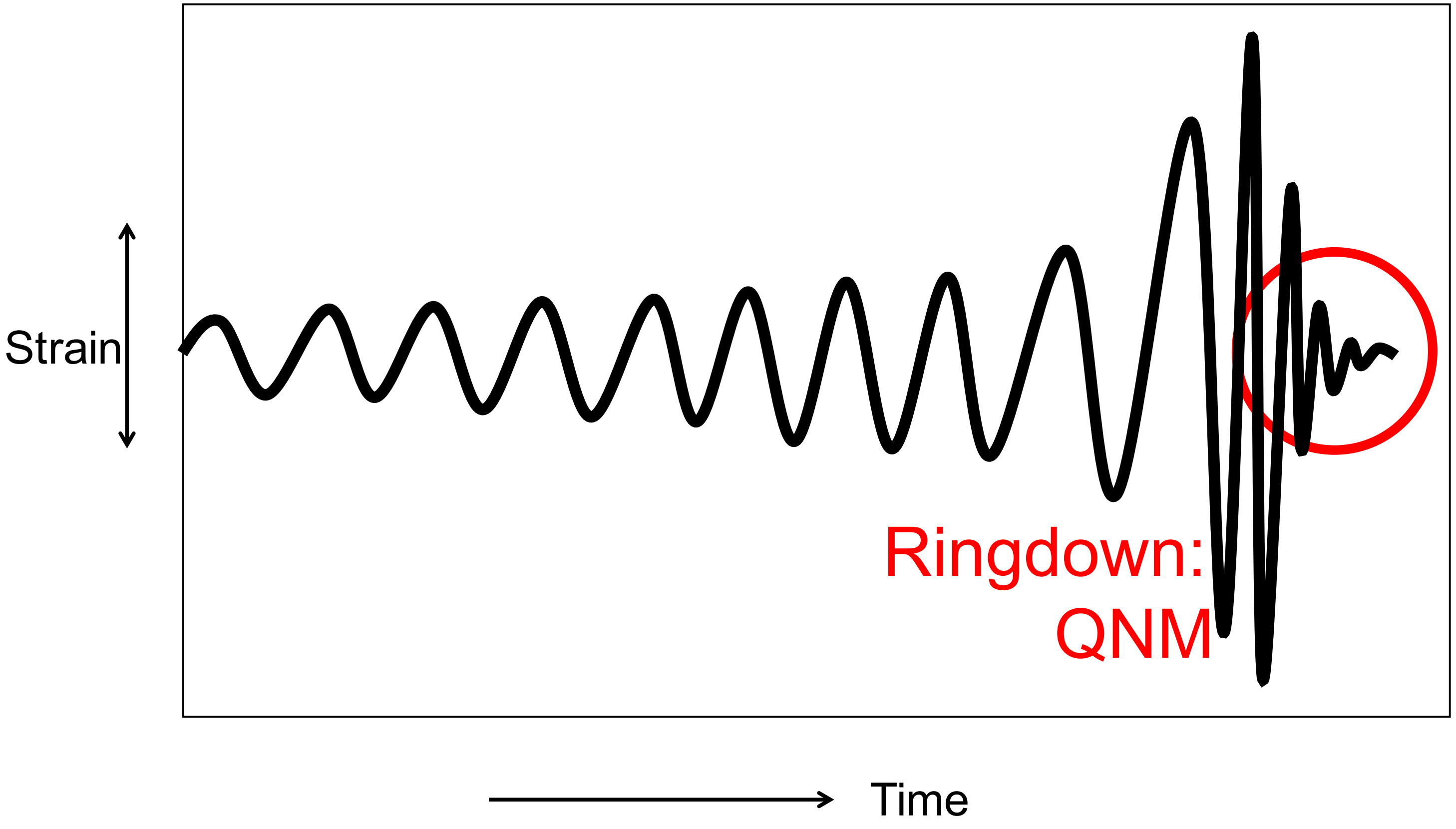
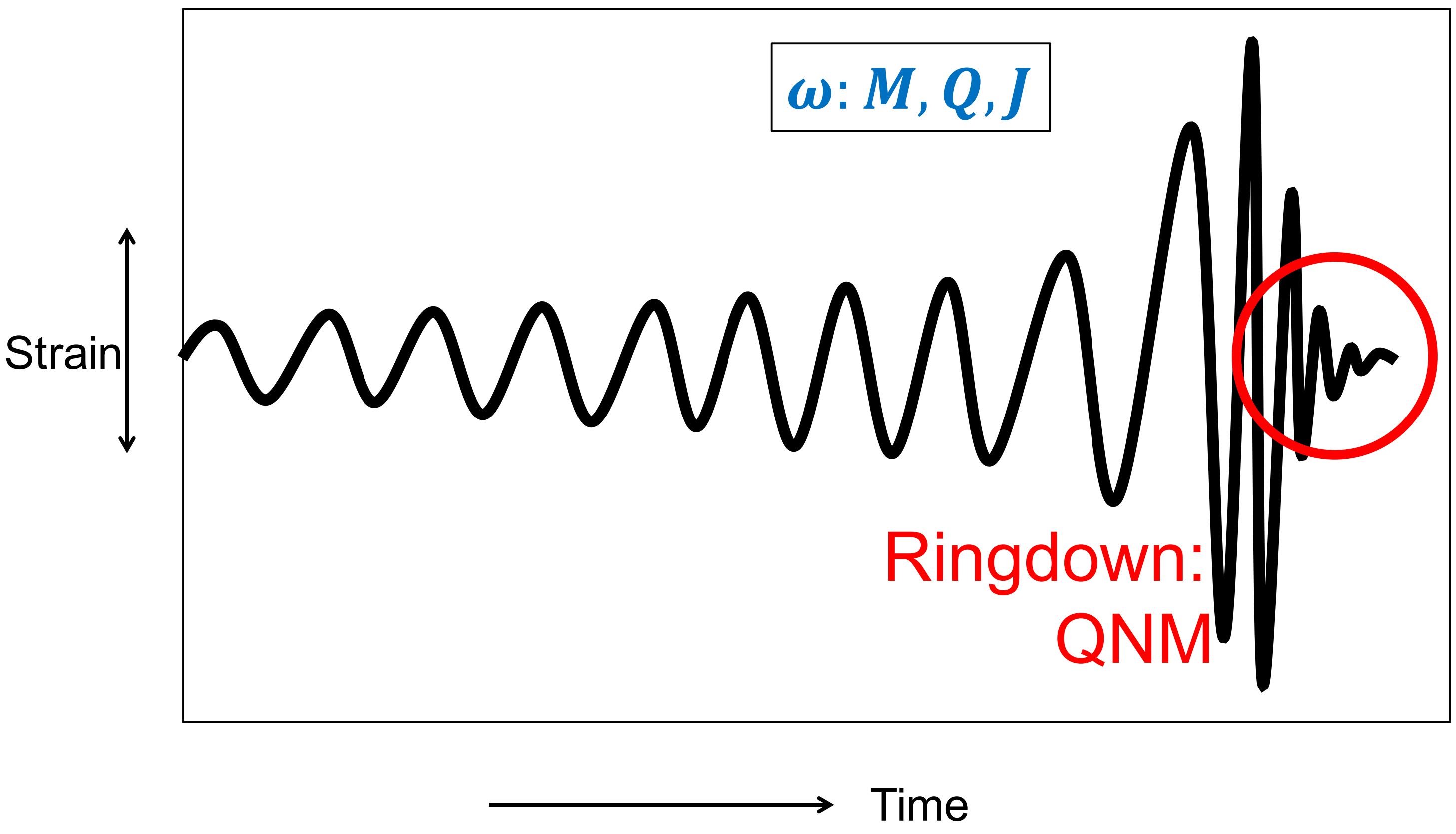
Black hole merger



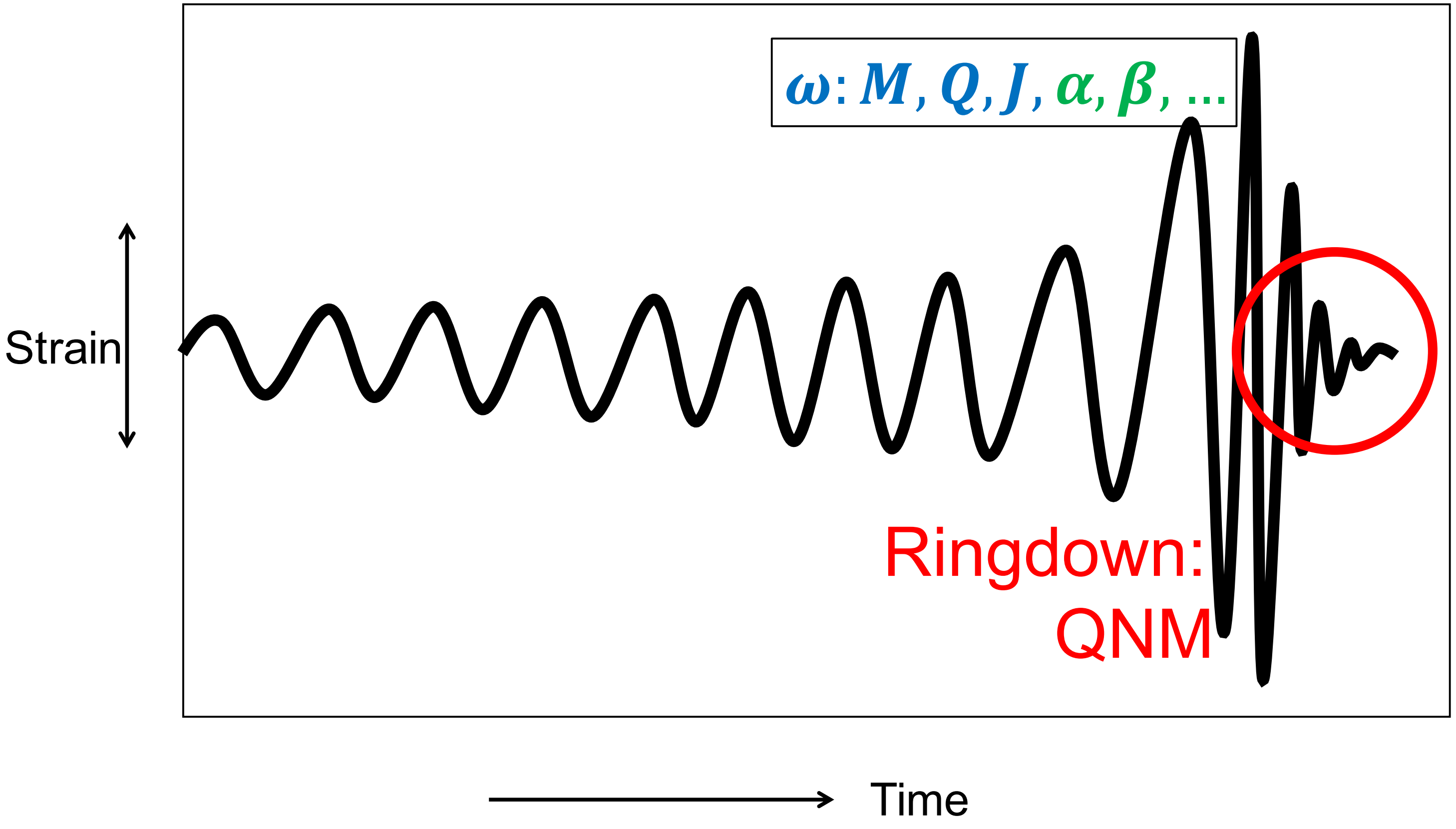
Quasinormal modes



Quasinormal modes



Black hole spectroscopy

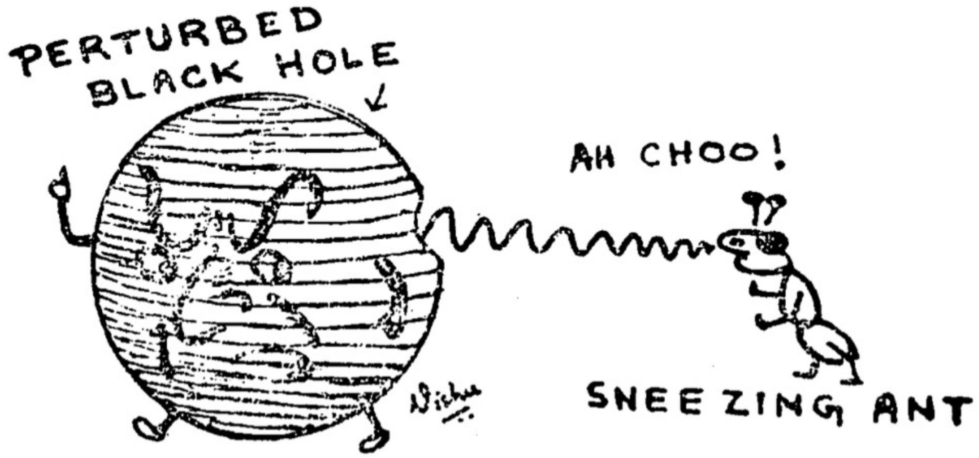


Black hole QNMs: Master equations

- Test fields in the black hole spacetime
  - Scalar field perturbations: KG equation
  - EM perturbations: Maxwell equation

- Gravitational perturbations

$$g_{\mu\nu} = g_{0\mu\nu} + h_{\mu\nu} \qquad |h| \ll 1$$



Black hole perturbations: Master equations

- $g_{\mu\nu} = g_{0\mu\nu} + h_{\mu\nu}$ in Regge-Wheeler gauge and after Fourier decom.:

Odd parity (axial)

$$\tilde{h}_{\mu\nu} = \begin{bmatrix} 0 & 0 & 0 & h_0(r) \\ 0 & 0 & 0 & h_1(r) \\ 0 & 0 & 0 & 0 \\ h_0(r) & h_1(r) & 0 & 0 \end{bmatrix} \left(\sin\theta \frac{\partial}{\partial\theta}\right) Y_{l0}(\theta),$$

Even parity (polar)

$$\tilde{h}_{\mu\nu} = \begin{bmatrix} H_0(r) f & H_1(r) & 0 & 0 \\ H_1(r) & H_2(r)/f & 0 & 0 \\ 0 & 0 & r^2 K(r) & 0 \\ 0 & 0 & 0 & r^2 K(r) \sin^2\theta \end{bmatrix} Y_{l0}(\theta).$$

Black hole perturbations: Master equations

$$\left(\frac{d^2}{dr_*^2}+\omega^2\right)\Psi = V_{a/p}\Psi$$

- Schwarzschild black hole:

  - Odd parity (axial): Regge-Wheeler equation

$$V_a = \left(1-\frac{2M}{r}\right)\left[\frac{l(l+1)}{r^2}-\frac{6M}{r^3}\right]$$

  - Even parity (polar): Zerilli equation

$$V_p = \frac{2\left(1-\frac{2M}{r}\right)\left[\lambda^2(\lambda+1)r^3+3M\lambda^2r^2+9M^2\lambda r+9M^3\right]}{r^3(\lambda r+3M)^2}$$

$$\lambda = (l+2)(l-1)/2$$

- In test field scenarios, the master equation can also be written in the Schrödinger-like form

Black hole perturbations: Master equations

- Kerr black hole: Teukolsky equation
- Newman-Penrose formalism – (l, n, m, m^*)
- Described by Weyl scalars (some contractions of $C_{\alpha\beta\gamma\rho}$)

Spin-weighted spheroidal harmonics

Angular eq. →

$$\left[\frac{\partial}{\partial u}(1-u^2)\frac{\partial}{\partial u}\right]{}_sS_{lm} + \left[a^2\omega^2u^2 - 2a\omega su + s + {}_sA_{lm} - \frac{(m+su)^2}{1-u^2}\right]{}_sS_{lm} = 0,$$

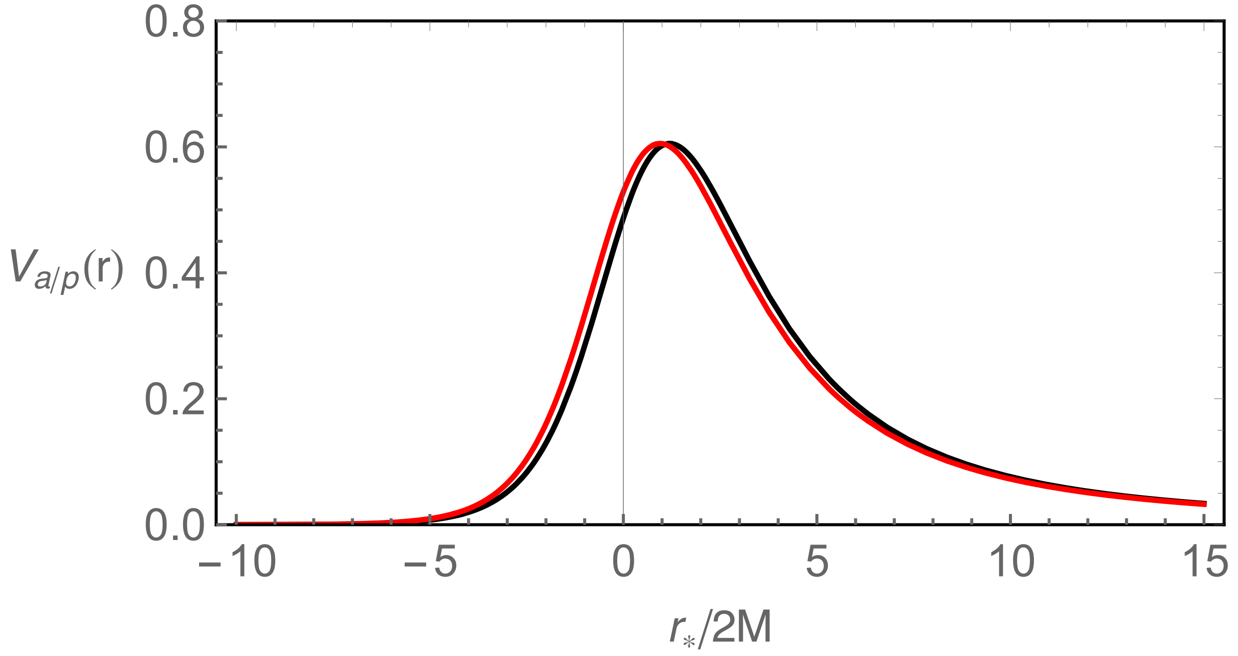
Radial eq. →

$$\Delta\partial_r^2 R_{lm} + (s+1)(2r-2M)\partial_r R_{lm} + V R_{lm} = 0.$$

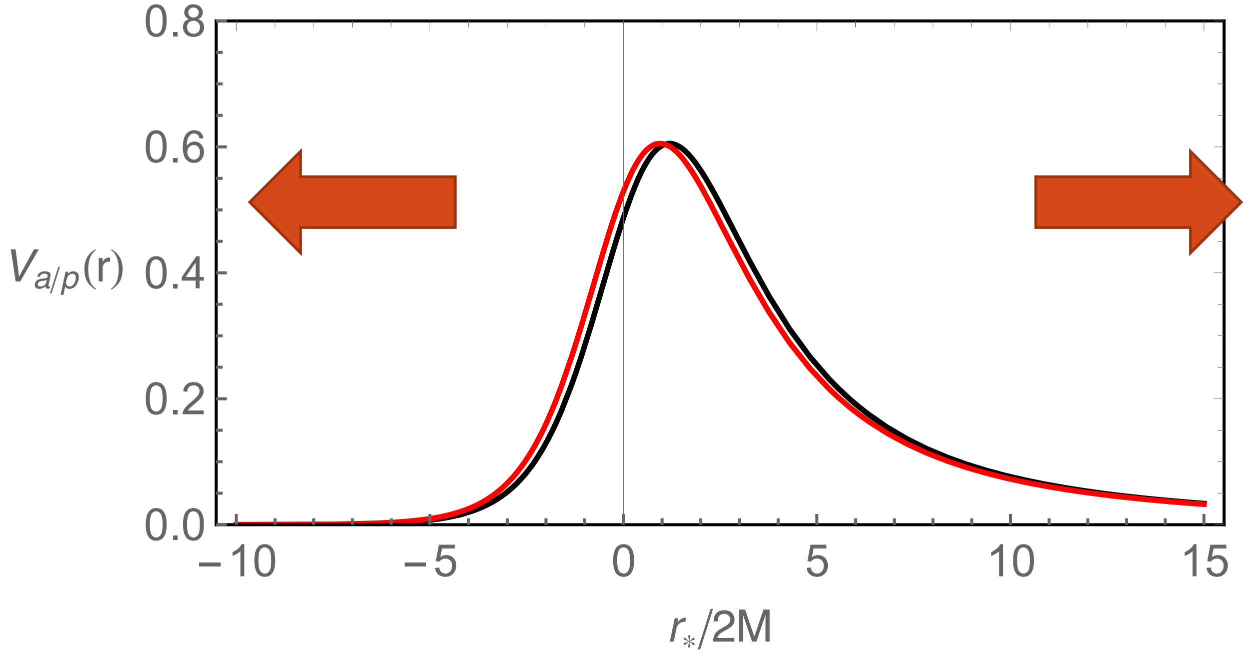
Here $u \equiv \cos\theta$, $\Delta = (r-r_-)(r-r_+)$ and

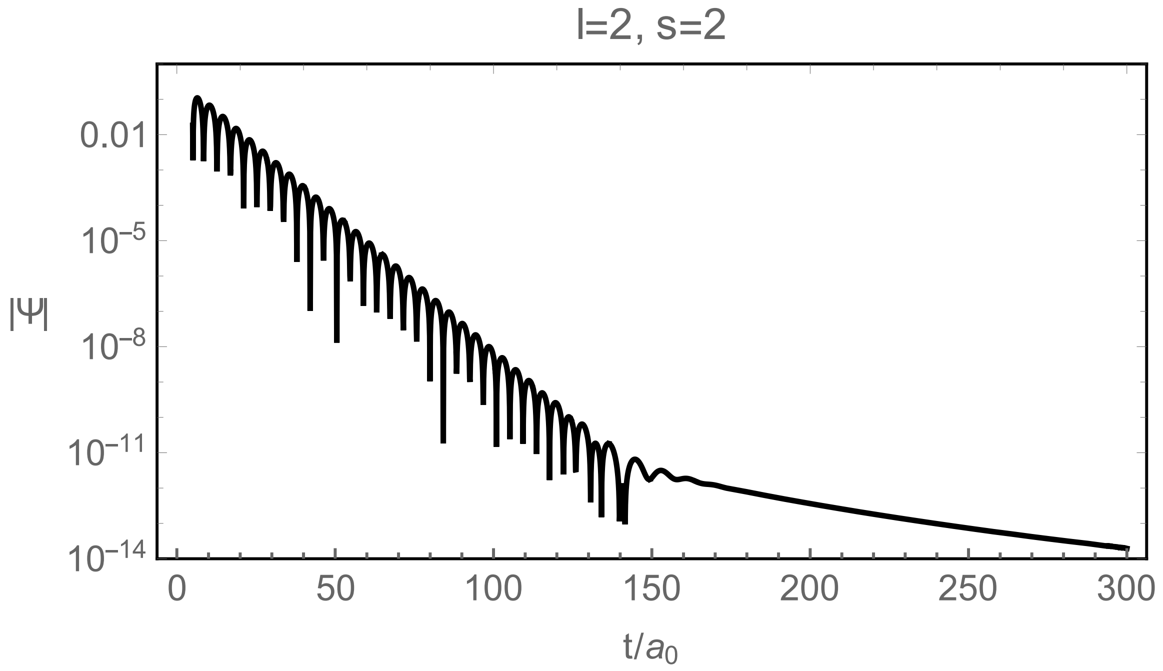
$$V = 2is\omega r - a^2\omega^2 - {}_sA_{lm} + \frac{1}{\Delta}[(r^2+a^2)^2\omega^2 - 4Mam\omega r + a^2m^2 + is(am(2r-2M) - 2M\omega(r^2-a^2))].$$

Potential and boundary conditions

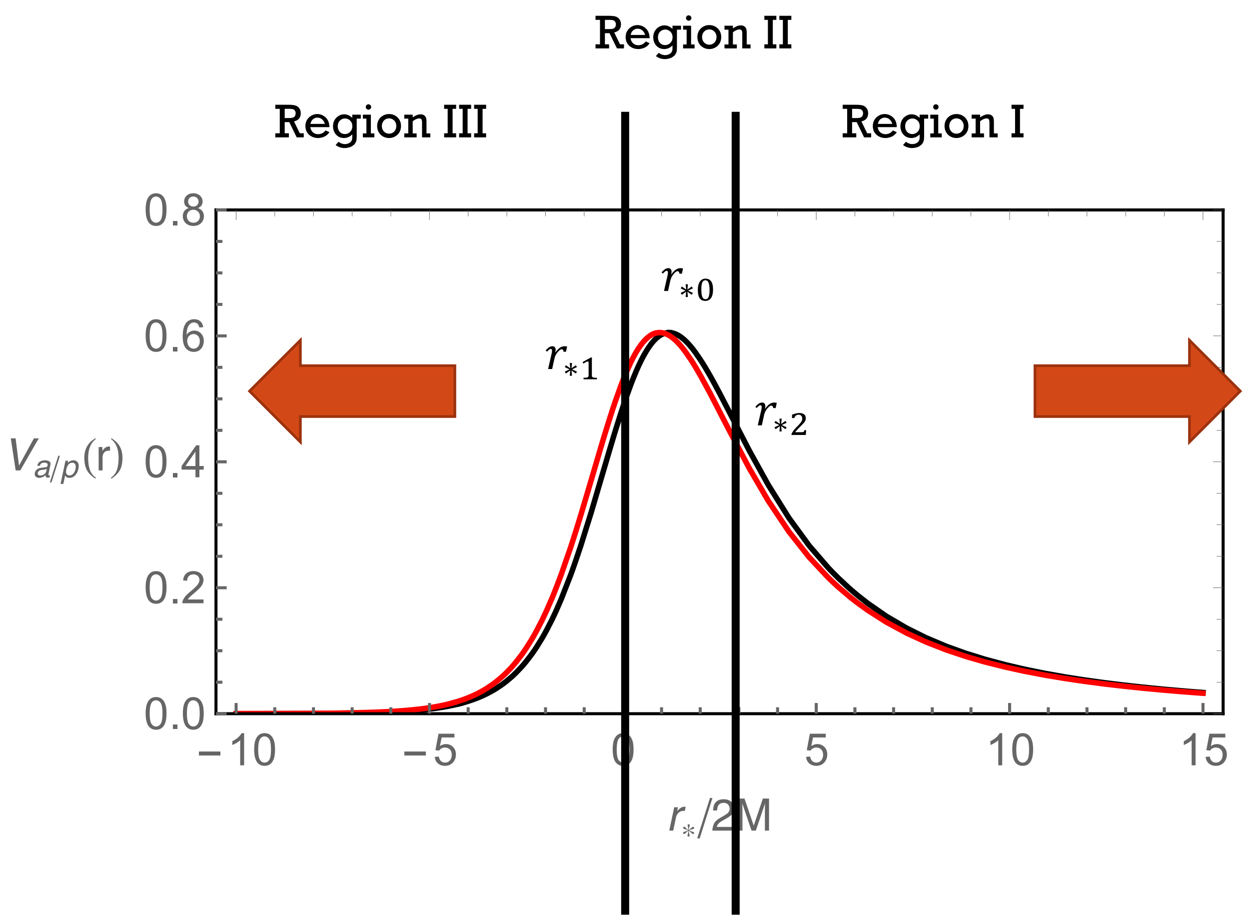


Potential and boundary conditions





WKB method for QNM calculations



$$\left(\frac{d^2}{dr_*^2} + \omega^2\right)\Psi = V_{a/p}\Psi$$

$$Q(r_*) \equiv \omega^2 - V_{a/p}$$

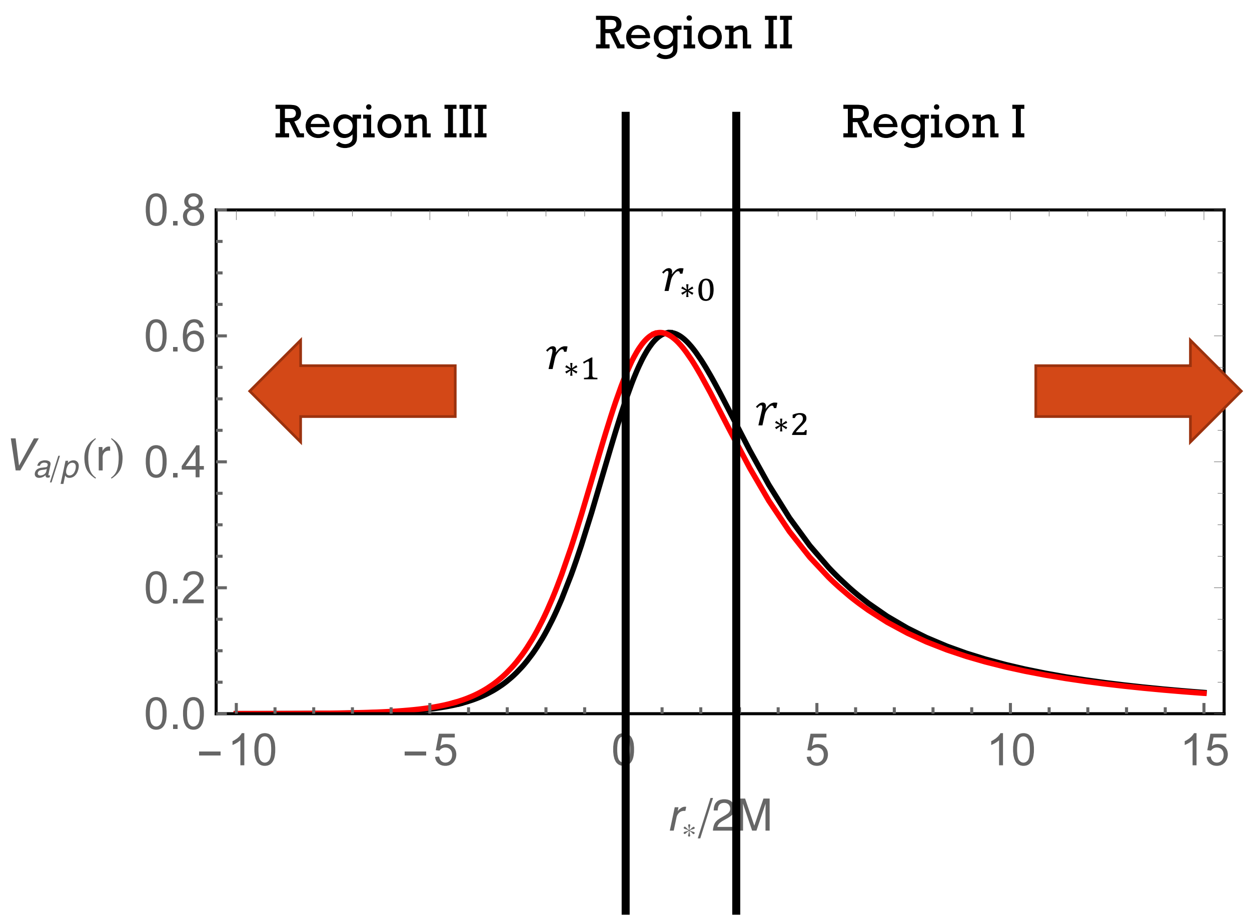
- Scatterint problem with zero incident waves
- $|Q_{max}| \ll |Q(\pm\infty)|$

$$\Psi_I(r_*) \approx Q^{-\frac{1}{4}} \exp\{-i \int_{r_{*2}}^{r_*} [Q(t)]^{\frac{1}{2}} dt\}$$

$$\Psi_{III}(r_*) \approx Q^{-\frac{1}{4}} \exp\{-i \int_{r_*}^{r_{*1}} [Q(t)]^{\frac{1}{2}} dt\}$$

Schutz, Will (1985)

WKB method for QNM calculations



$$\Psi_I(r_*) \approx Q^{-\frac{1}{4}} \exp\{-i \int_{r_{*2}}^{r_*} [Q(t)]^{\frac{1}{2}} dt\}$$

$$\Psi_{III}(r_*) \approx Q^{-\frac{1}{4}} \exp\{-i \int_{r_*}^{r_{*1}} [Q(t)]^{\frac{1}{2}} dt\}$$

Approximating Q in region II as a parabola, the solution is given by

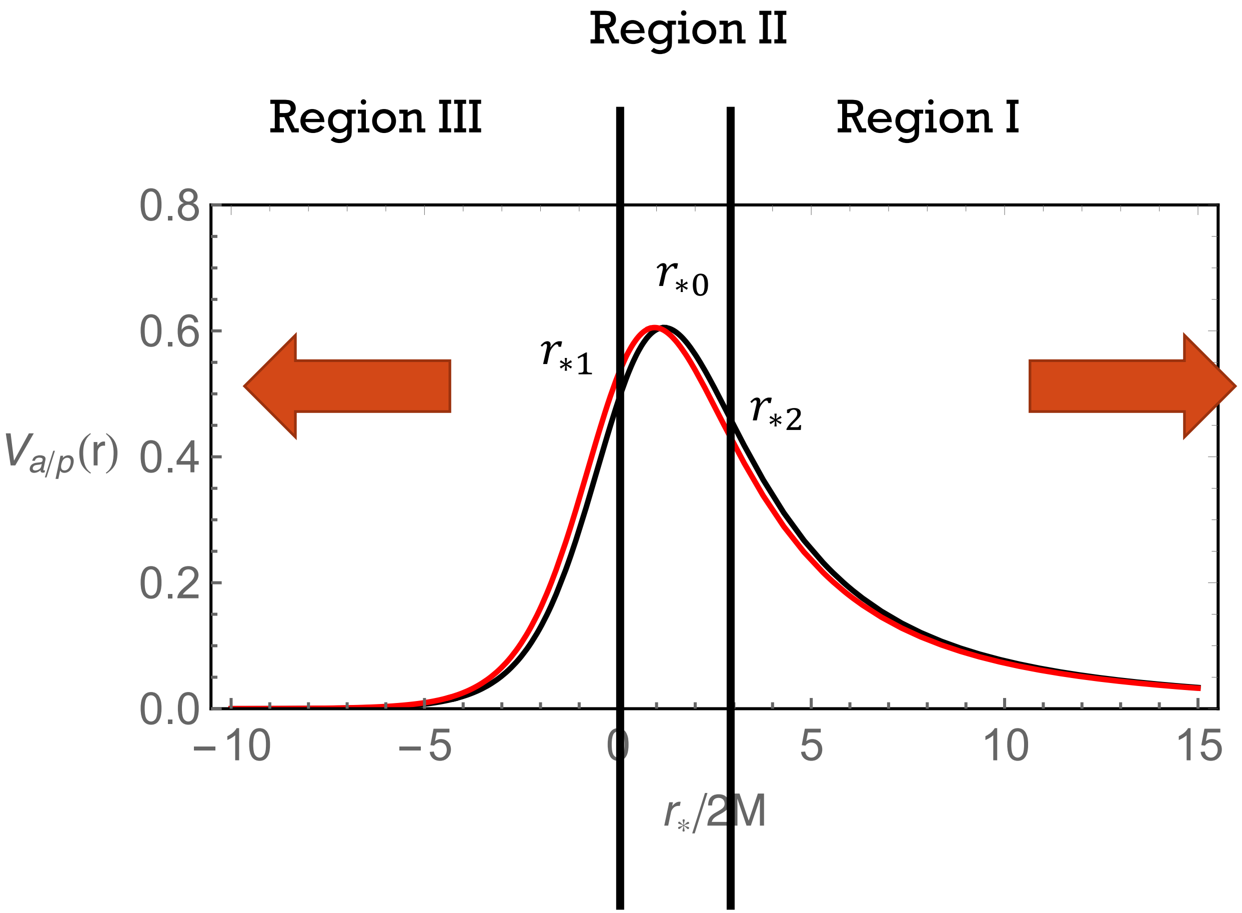
$$\Psi_{II}(x) \approx c_1 D_\nu(x) + c_2 D_{-\nu-1}(ix)$$

where

$x \equiv (2Q_0'')^{\frac{1}{4}} e^{\frac{i\pi}{4}} (r_* - r_{*0})$ and $\nu + \frac{1}{2} \equiv -iQ_0/(2Q_0'')^{1/2}$

Schutz, Will (1985)

WKB method for QNM calculations



Taking $x \to \pm\infty$ for Ψ_{II}, and conecting with Ψ_I Ψ_{III}, one gets $\Gamma(-\nu) = \infty$

which implies

$$\frac{Q_0}{(2Q_0'')^{\frac{1}{2}}} = i\left(n + \frac{1}{2}\right)$$

with $n = 0{,}1{,}2 \ldots$

WKB method for QNM calculations

- 1st order: Schutz, Will (1985)
- 3rd order: Iyer, Will (1987)
- 6th order: Konoplya (2003)
- Higher order: Matyjasek, Opala, Telecka (2017)(2019)
- Review: Konoplya, Zhidenko, Zinhailo (2019)

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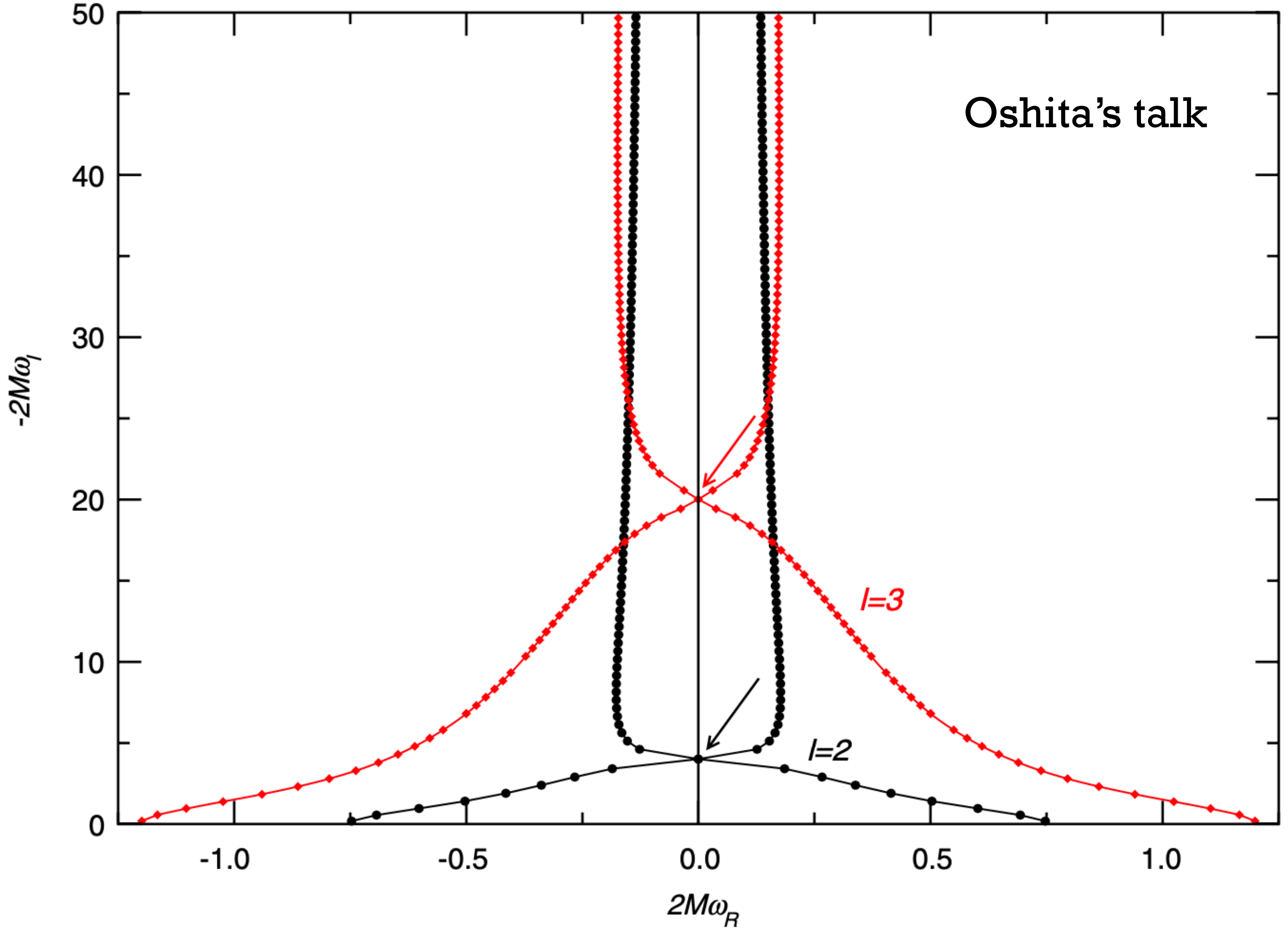
$$\frac{Q_0}{(2Q_0'')^{\frac{1}{2}}} = i\left(n + \frac{1}{2}\right)$$

with $n = 0,1,2\ldots$

The n-th order WKB method requires 2n-th order derivatives at the potential peak

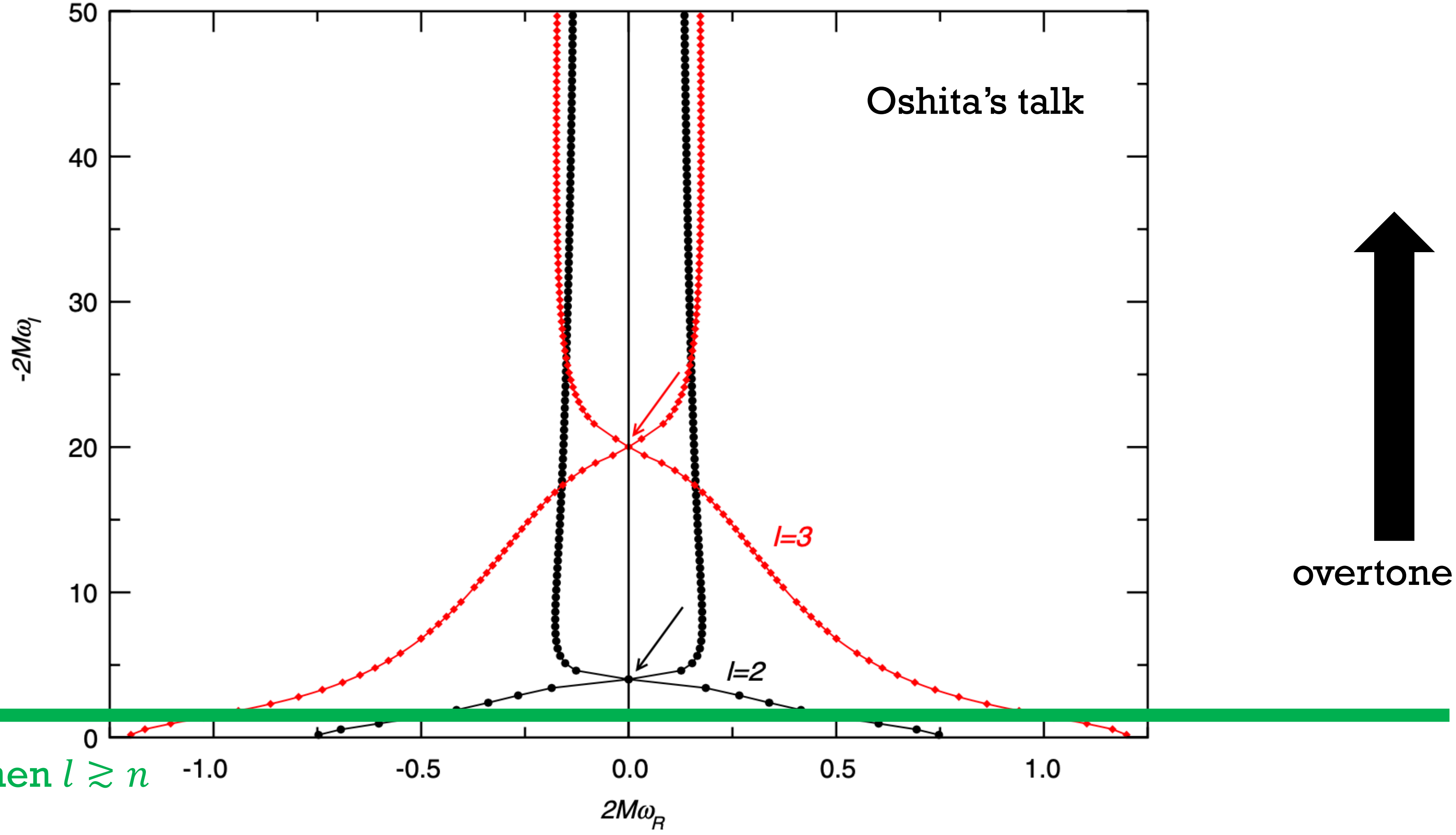
QNM spectrum



overtone: n

Berti, Cardoso, Starinets (2009)

QNM spectrum



overtone: n

WKB valid when $l \gtrsim n$

Berti, Cardoso, Starinets (2009)

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  - Basis
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- Black hole QNMs & WKB method

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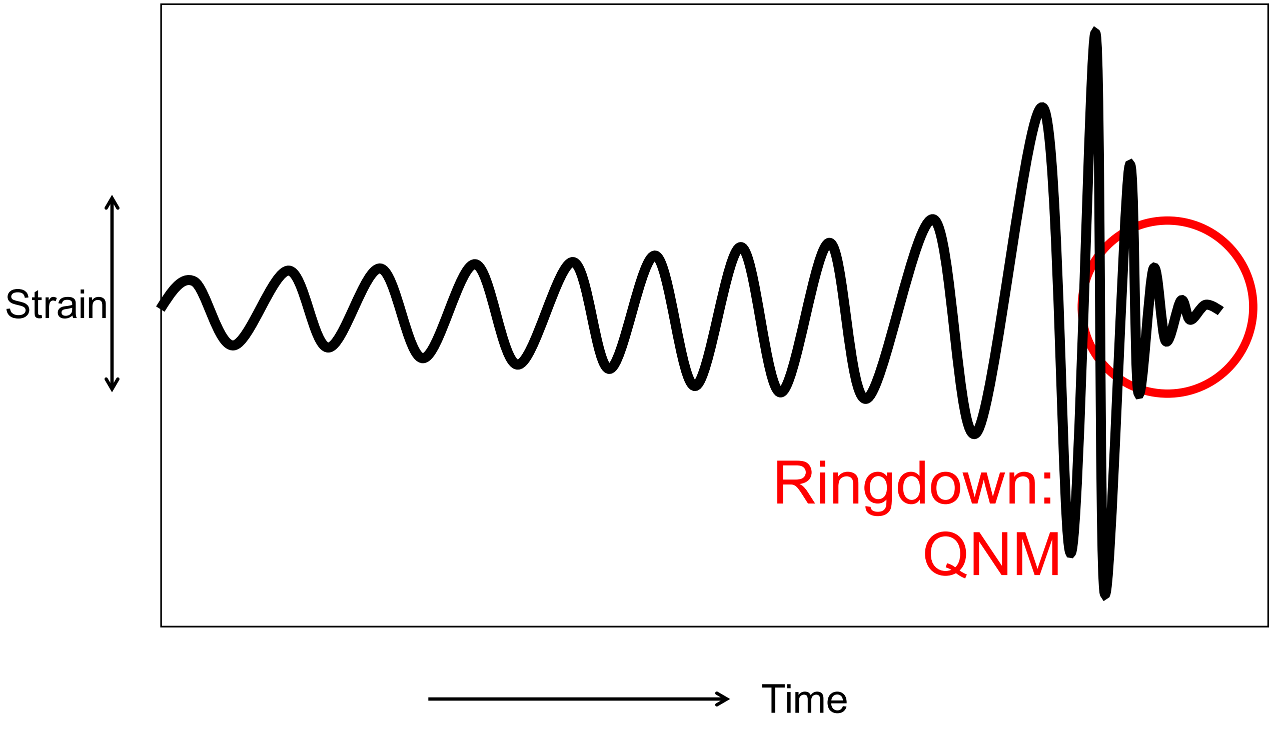
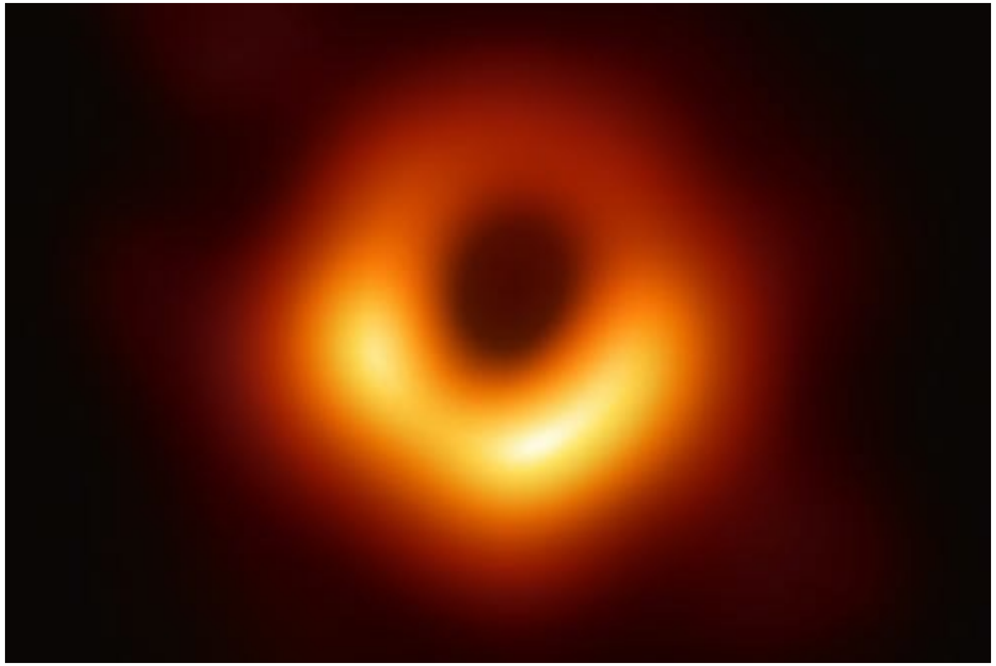
Black hole QNMs and Images





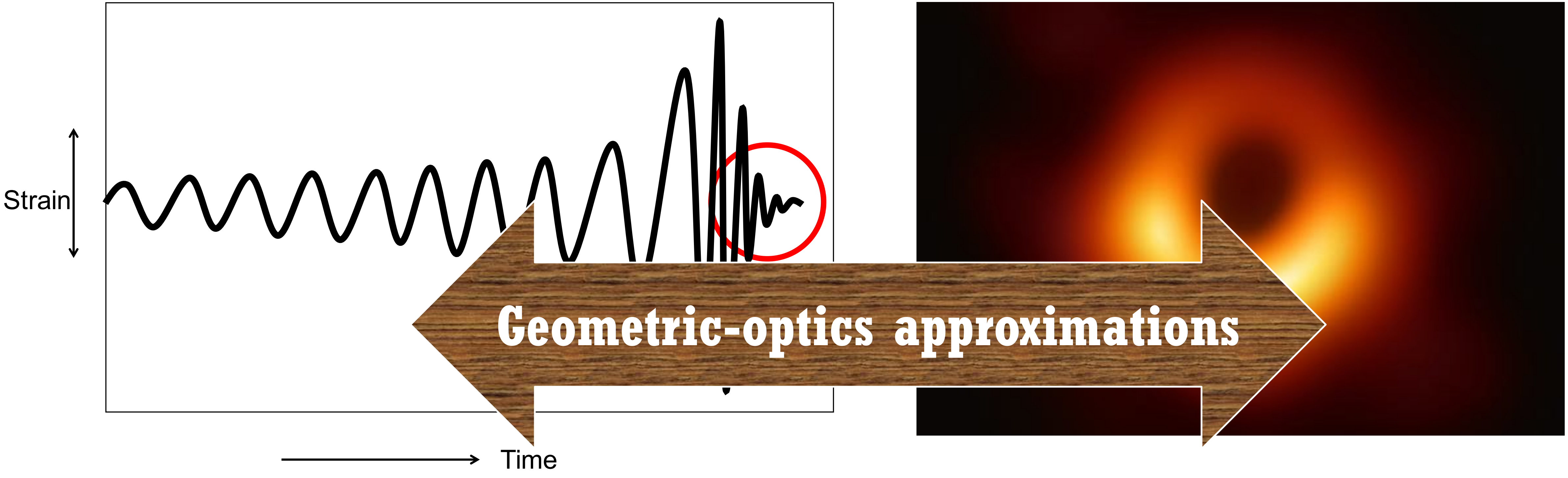
Field propagation in BH spacetimes

$$\nabla^{\alpha}\nabla_{\alpha}A = \cdots$$

Photon propagation in BH spacetimes

$$k^{\alpha}k_{\alpha} = 0$$

Geometric-Optics (Eikonal) Approximations



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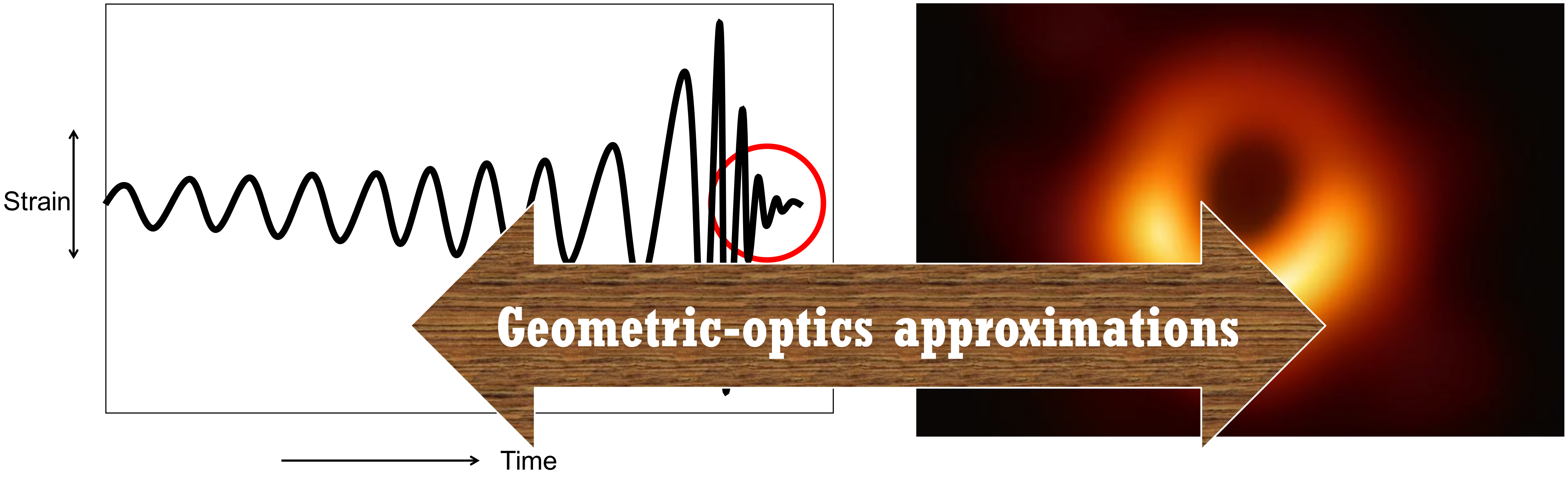
Geometric-Optics (Eikonal) Approximations



Field propagation in BH spacetimes

$$\nabla^{\alpha}\nabla_{\alpha}A = \boldsymbol{O(\lambda/L)} \sim \mathbf{0}$$

Photon propagation in BH spacetimes

$$k^{\alpha}k_{\alpha} = 0$$

How does the correspondence manifest in BH spacetimes?

- Schwarzschild BH:

- Odd parity (axial): Regge-Wheeler equation

$$V_a = \left(1 - \frac{2M}{r}\right)\left[\frac{l(l+1)}{r^2} - \frac{6M}{r^3}\right]$$

- Even parity (polar): Zerilli equation

$$V_p = \frac{2\left(1-\frac{2M}{r}\right)\left[\lambda^2(\lambda+1)r^3+3M\lambda^2r^2+9M^2\lambda r+9M^3\right]}{r^3(\lambda r+3M)^2}$$

$$\lambda = (l+2)(l-1)/2$$

- eikonal ($l \to \infty$) QNMs
- $V_a \sim V_p \sim \frac{l^2}{r^2}\left(1 - \frac{2M}{r}\right)$

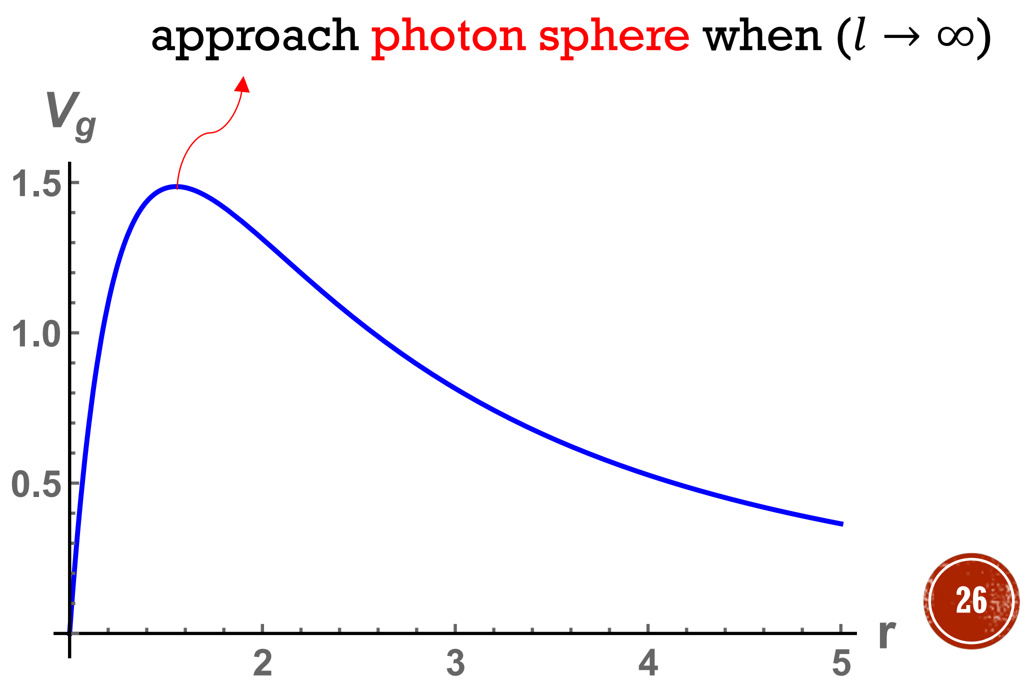
How does the correspondence manifest in BH spacetimes?

- Spacetime symmetry is crucial
- Non-rotating BH:

Static and spherically symmetric $ds^2 = -A(r)dt^2 + \frac{dr^2}{B(r)} + r^2 d\Omega^2$

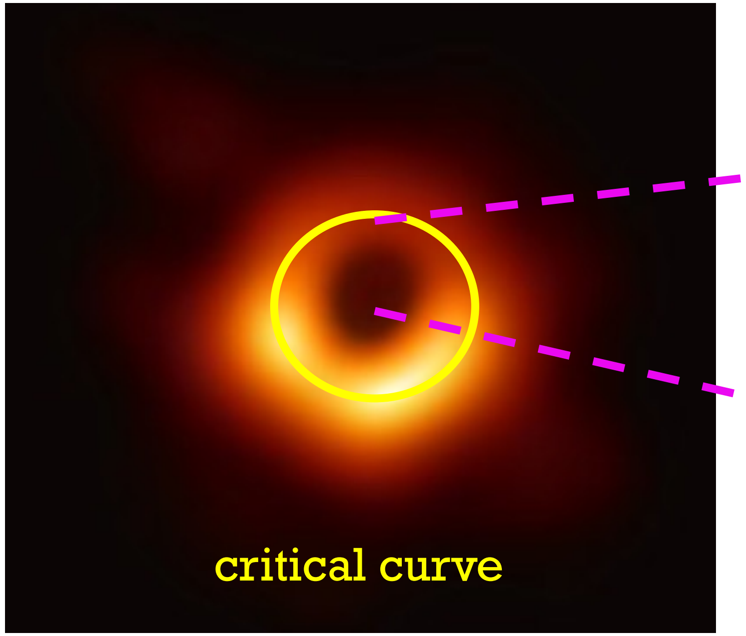
$$\left(\frac{d^2}{dr_*^2} + \omega^2\right)\Psi = V_g \Psi$$

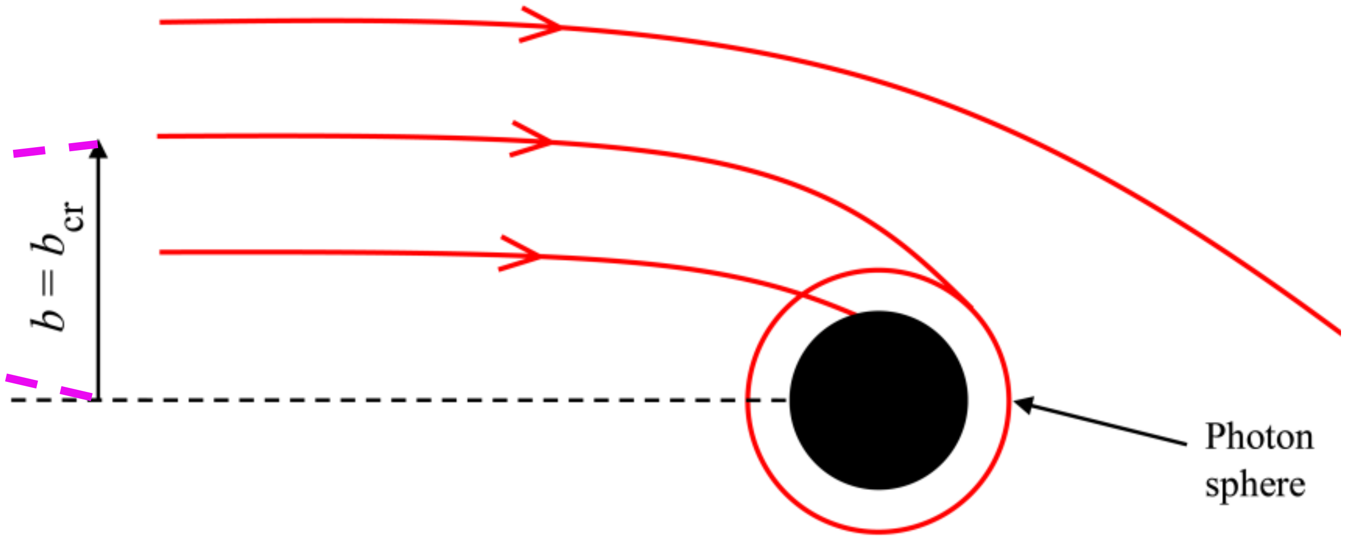
- The potential for eikonal ($l \to \infty$) QNMs: $V \approx \frac{A(r)}{r^2} l^2$
- The peak of the potential coincides with the photon sphere
  - Photon sphere equation: $\partial_r [A(r)/r^2] = 0$



Eikonal QNMs Correspondence

- The eikonal QNMs ($l \to \infty$) and the photon sphere





$$\omega \approx \Omega_c l - i(n + 1/2)|\lambda_c|$$

- $Re(\omega) \to \Omega_c$ (orbital frequency of the photon sphere)
- $Im(\omega) \to \lambda_c$ (Lyapunov exponent)
- $\gamma \equiv \lambda_c / \Omega_c$ (critical exponent)

Cardoso, Miranda, Berti, Witek, Zanchin (2009)

Correspondence in Kerr Spacetime

- Separable geodesic equations (Carter constant), and separable wave equations

Wave Quantity	Ray Quantity	Interpretation
ω_R	$\mathscr{E}$	Wave frequency is same as energy of null ray (determined by spherical photon orbit).
m	L_z	Azimuthal quantum number corresponds to z angular momentum (quantized to get standing wave in ϕ direction).
A^R_{lm}	$\mathscr{Q}+L_z^2$	Real part of angular eigenvalue related to Carter constant (quantized to get standing wave in θ direction).
ω_I	$\gamma=-\mathscr{E}_I$	Wave decay rate is proportional to Lyapunov exponent of rays neighboring the light sphere.
A^I_{lm}	$\mathscr{Q}_I$	Nonzero because $\omega_I \neq 0$ (see Secs. II B 2 and III C 3 for further discussion).

Yang *et al*. (2012)

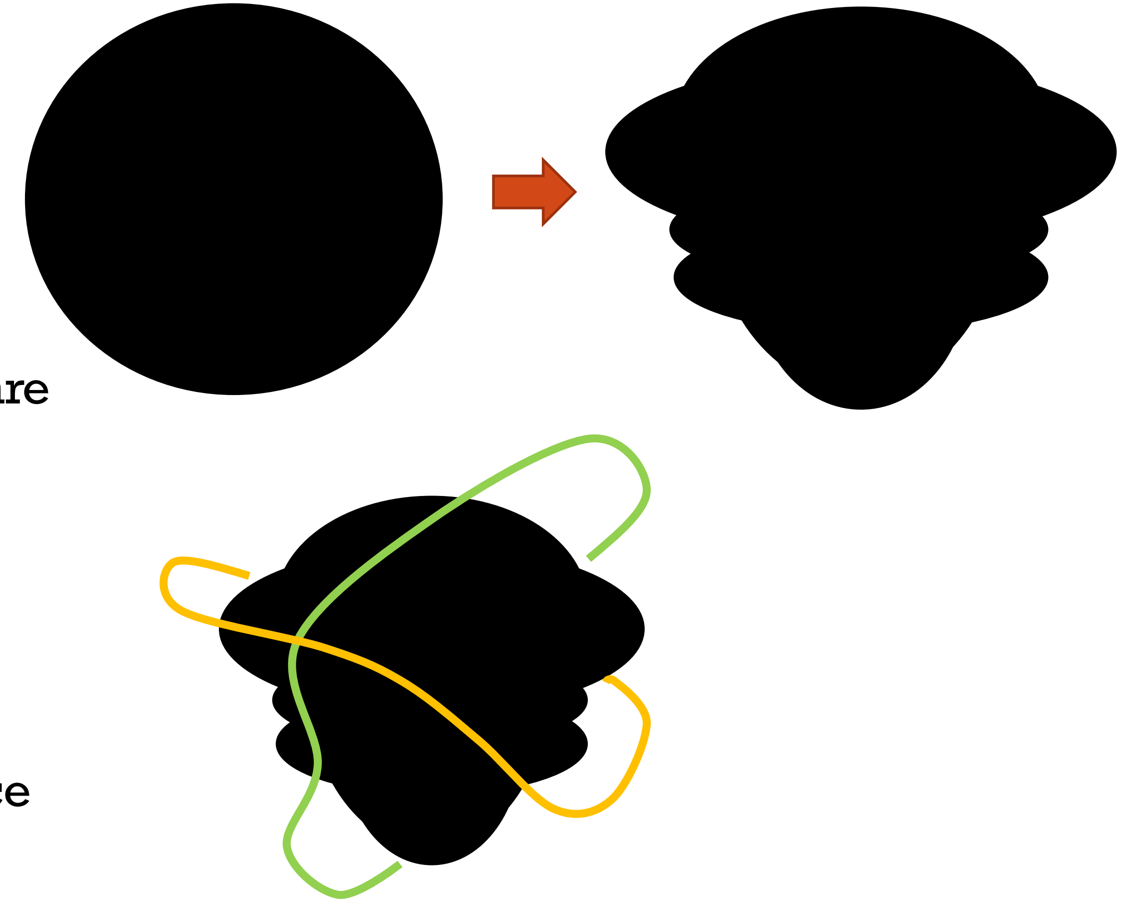
Recently extended to Kerr-Newman by Li *et al*. (2021)

Correspondence in deformed BH spacetimes

- Consider general axisymmetric deformations of Schwarzschild BHs

In the presence of deformations:

- Radial and latitudinal sectors of geodesic equations are NOT separable
- Generic photon orbits $r(\theta)$ do NOT have constant r
- Complicated wave equations
- <u>If deformations are small</u>: Identify the correspondence by defining the averaged radius along full closed photon orbits



CYC, Chiang, Tsao (2022)

Correspondence in dynamical BH spacetimes

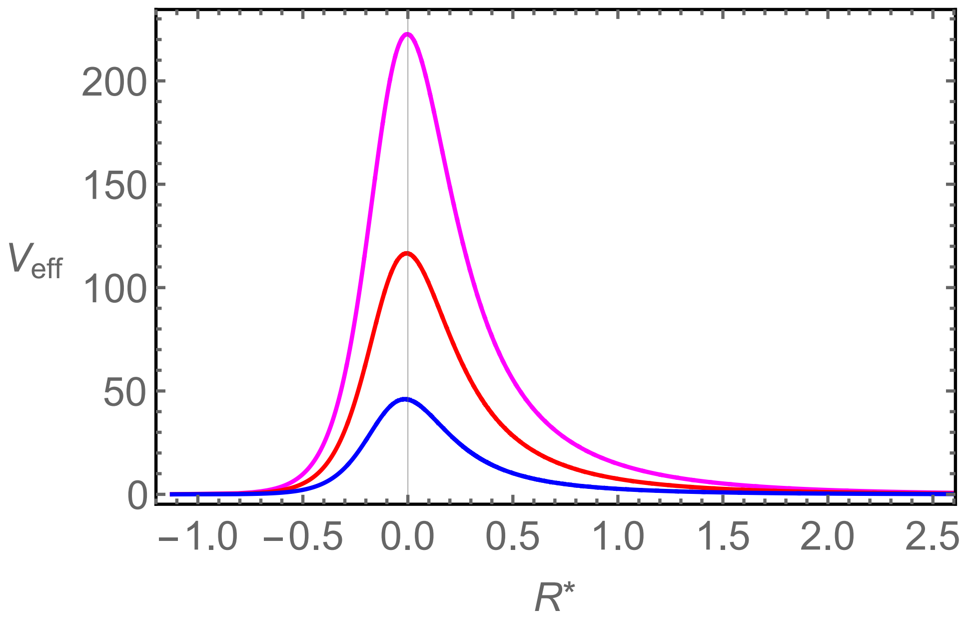
- Consider Vaidya spacetime

$$ds^2 = -\left(1 - \frac{2m(v)}{r}\right) dv^2 + 2drdv + r^2 d\Omega_2^2$$

- Generic mass function $m(v)$: only numerics

- Preliminary results on linear accretion $m(v) = \mu v$
    - Conformal Killing vector
    - Separable Klein-Gordon equation
    - Photon sphere uniquely defined

Solanki, Perlick (2022), Koga, Asaka, Kimura, Okabayashi (2022)



Capuano, Santoni, Barausse (2024), **CYC**, Chiang, Koga (ongoing)

Eikonal QNMs and BH Shadows

ω_R $\leftrightarrow$ Angular frequency on PS $\leftrightarrow$ Size of shadow image

ω_I $\leftrightarrow$ Lyapunov exponent on PS $\leftrightarrow$ Higher-order ring structures

Jusufi (2020), Cuadros-Melgar *et al*. (2020)

Jusufi (2020), Yang (2021)

- Can the eikonal correspondence be violated?
- Can it be tested observationally?

- Black hole QNMs & WKB method
- Eikonal correspondence between QNMs and shadows
  - Basis
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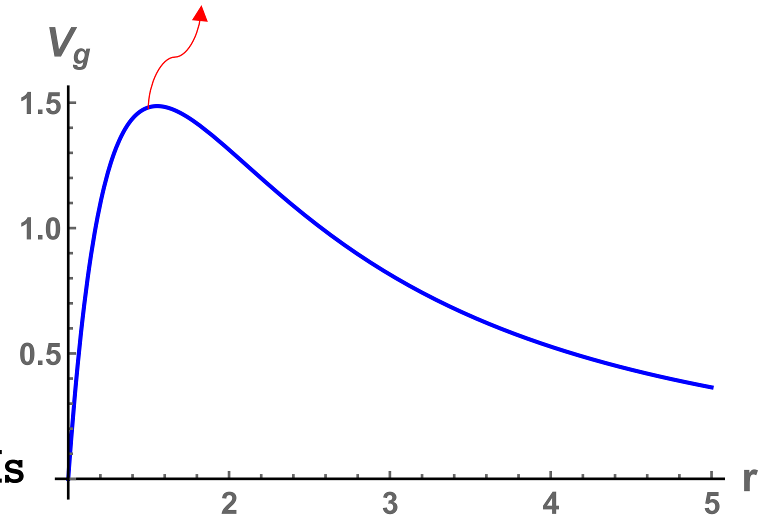
- Conclusions

Eikonal Correspondence Violation

- In GR, the potential for eikonal ($l \to \infty$) QNMs: $V \approx \frac{A(r)}{r^2} l^2$
- The peak of the potential coincides with the photon sphere
  - Photon sphere equation: $\partial_r [A(r)/r^2] = 0$



- This may not be true for modified gravity: $$V \approx \alpha(r) \left(\frac{A(r)}{r^2} l^2 \right)$$
- The peak of the potential may differ from the photon sphere of BHs
  - Non-minimal coupling between matter and curvature
  - String-inspired models

CYC, De Felice, Tsujikawa (2024) **CYC**, Bouhmadi-López, Chen (2019) (2021) **CYC**, Chen (2020)

Cardoso, Gualtieri (2010) Konoplya, Stuchlik (2017) Moura, Rodrigues (2021)

- A preliminary proposal (i.e., nonrotating BH) for testing eikonal correspondence based on joint observations of ringdown and image black holes with similar masses

QNM Observables

$$\gamma_l^{QNM} \equiv 2l \frac{|\omega_I|}{\omega_R}$$

l: multipole number

If the eikonal correspondence is satisfied:

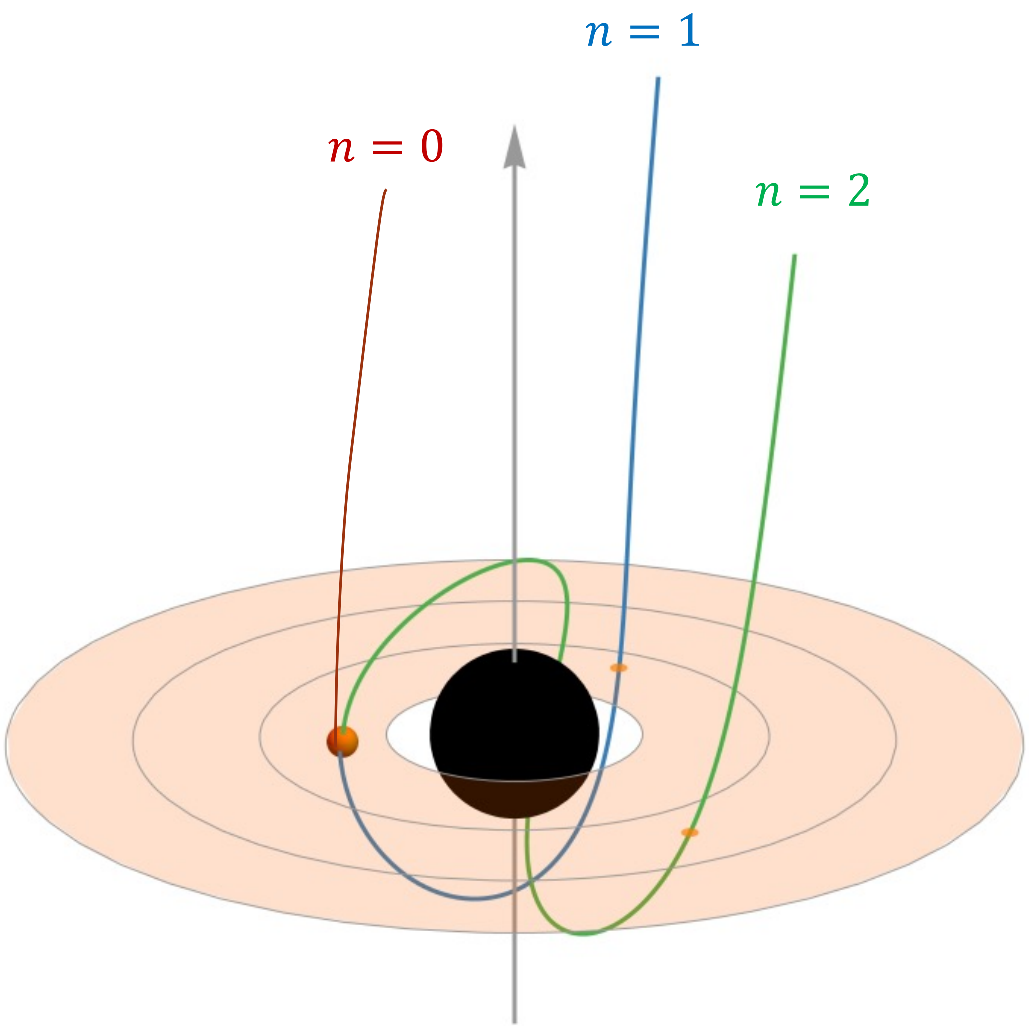
$$\gamma_l^{QNM} = \left(1 - \frac{1}{2l}\right)\gamma + O(l^{-2})$$

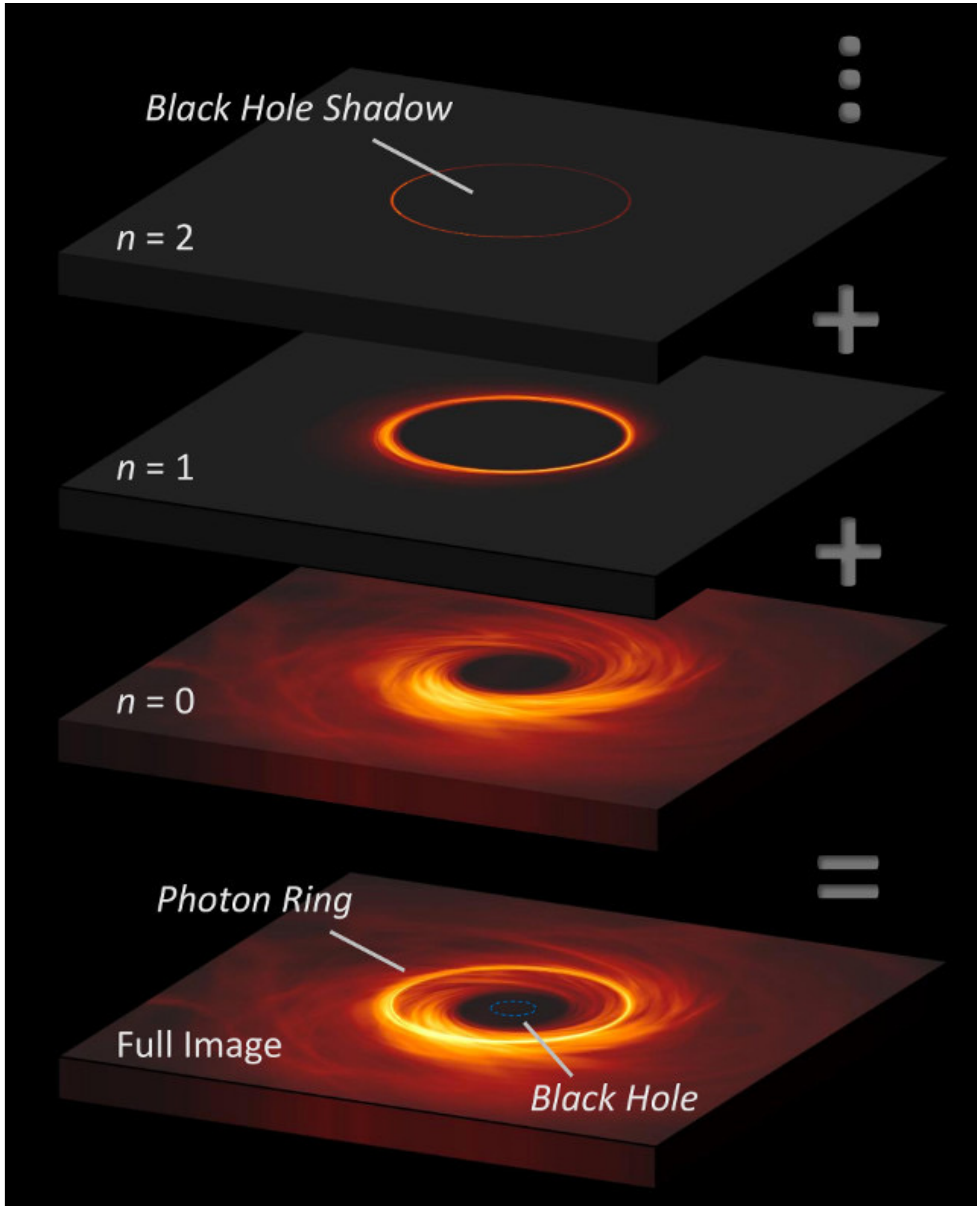
$\gamma \equiv \lambda_c/\Omega_c$ (critical exponent)

γ_l^{QNM} converges to γ from below when $l \to \infty$

Chen *et al.* (2022)

Photon Ring Observables





Gralla, et al. (2020)

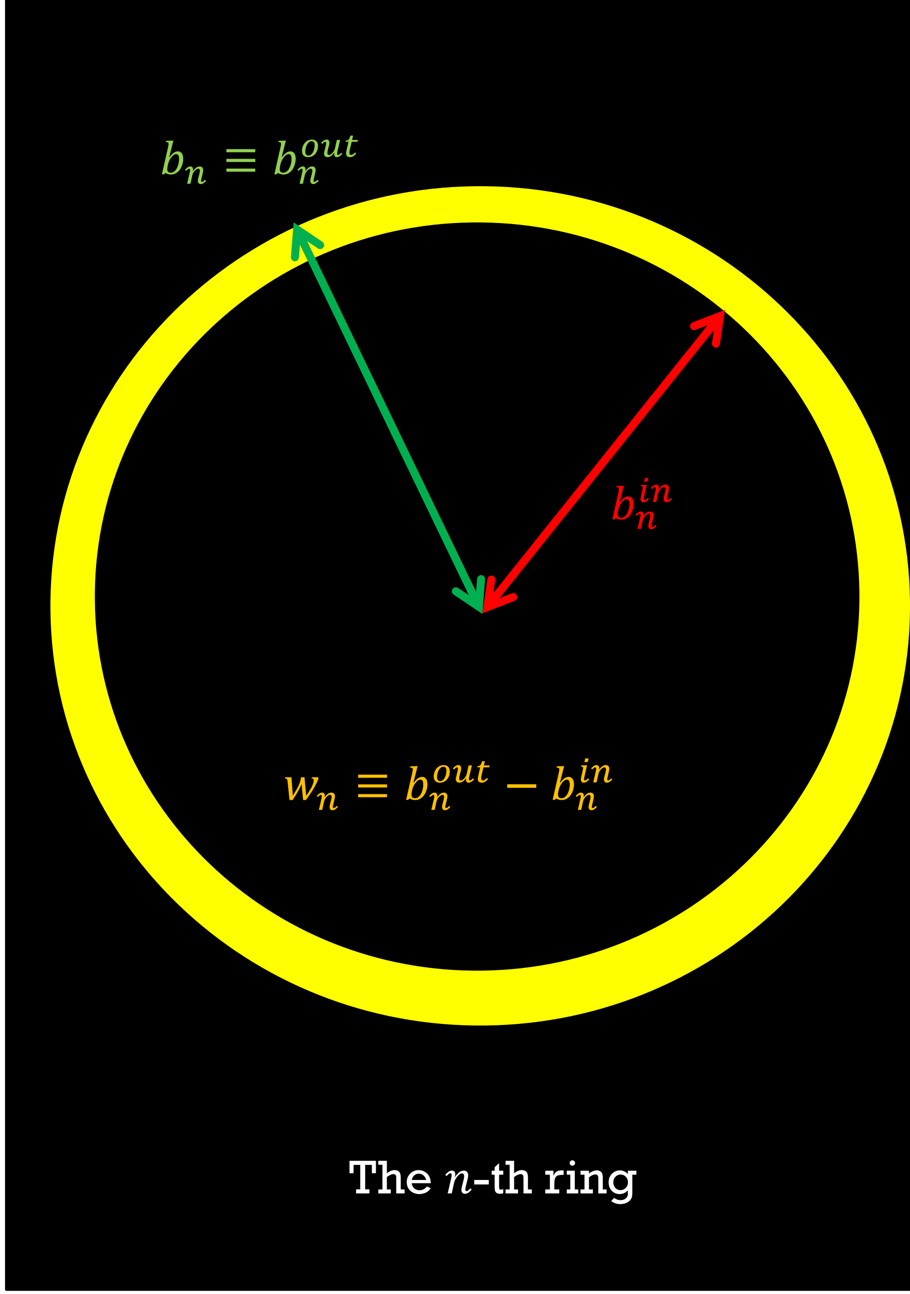
Photon Ring Observables

$$\gamma_n^w \equiv \frac{1}{\pi} \ln \frac{w_n}{w_{n+1}}$$

$$\gamma_n^b \equiv \frac{1}{\pi} \ln \frac{b_n - b_{n+1}}{b_{n+1} - b_{n+2}}$$

- Two ring observables converge to γ <u>from above</u> when $n \to \infty$

$\gamma \equiv \lambda_c/\Omega_c$ (critical exponent)



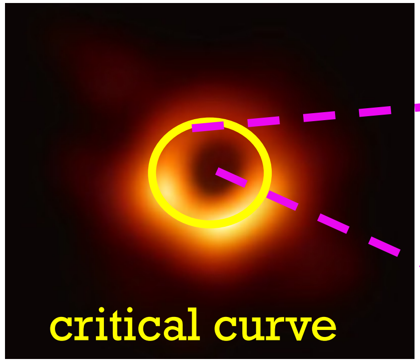
Chen *et al.* (2022)

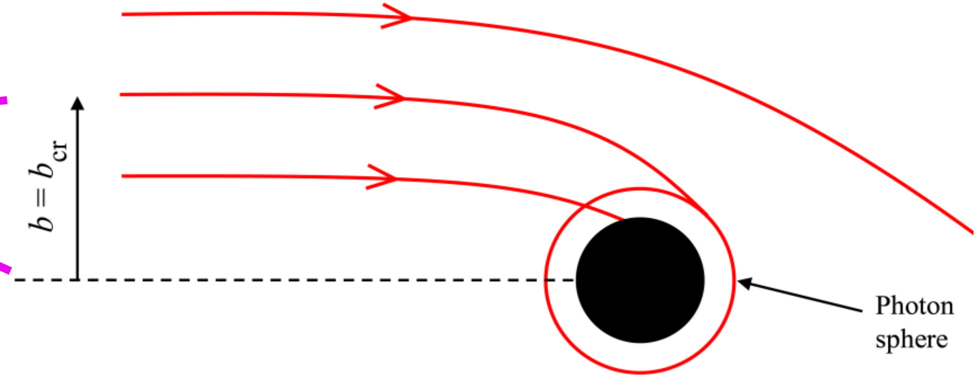
Photon Ring Observables

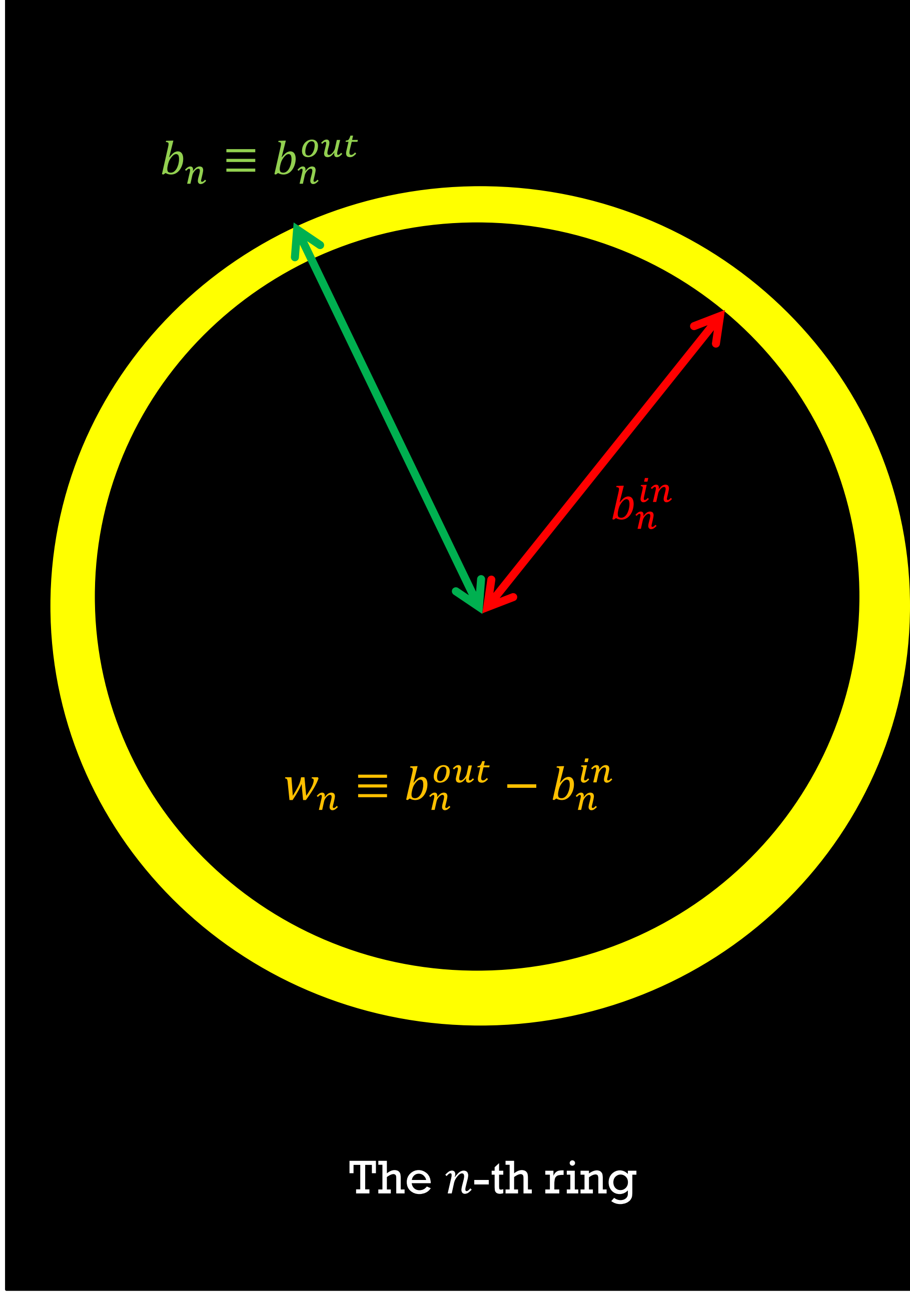
- Strong Lensing formula:

$$-\gamma\left(n+\frac{1}{2}\right)\pi = \ln a_n - Y(z_s, z_d) + \epsilon_n$$

  - $a_n = 1 - b_c/b_n \qquad \epsilon_n = C a_n \ln a_n$
  - Y: A constant determined by source and detector







Chen *et al*. (2022)

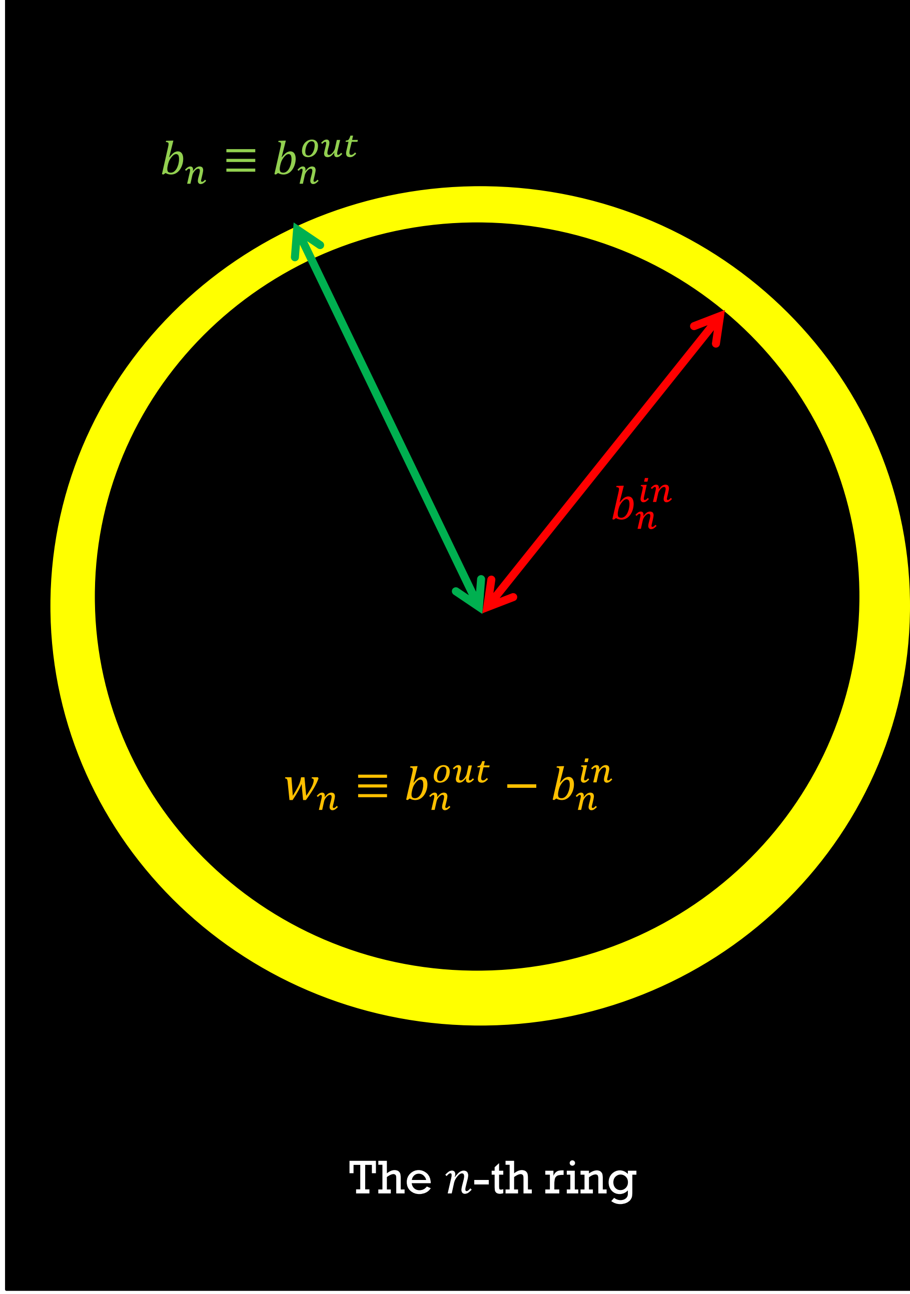
Photon Ring Observables

- $a_n = 1 - b_c/b_n \qquad \epsilon_n = C a_n \ln a_n$

$$\gamma_n^w \approx \gamma + (\epsilon_{n+1} - \epsilon_n)/\pi$$

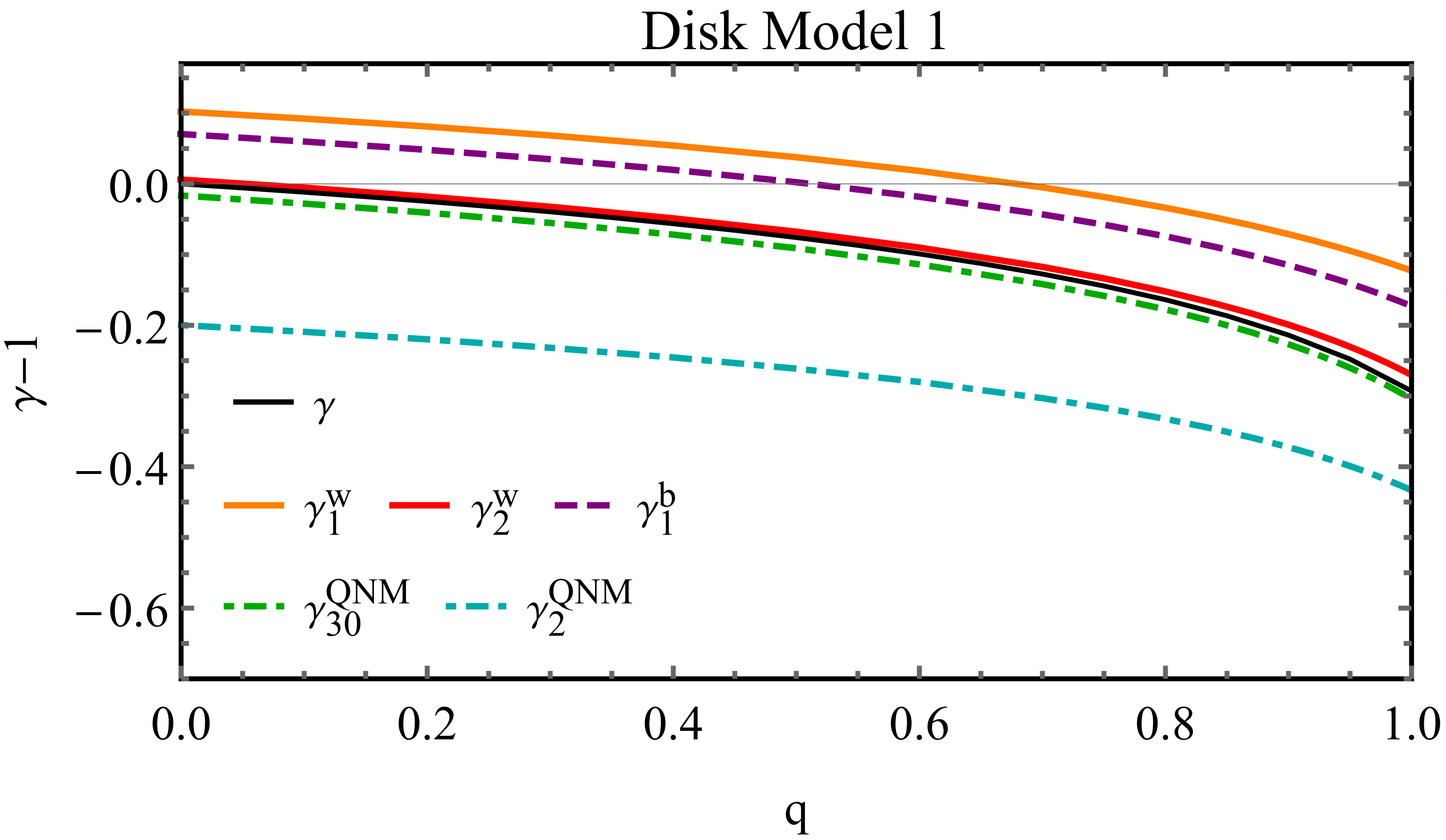
$$\gamma_n^b \approx \gamma + \frac{\epsilon_{n+1} - \epsilon_n - e^{-\gamma\pi}(\epsilon_{n+2} - \epsilon_{n+1})}{\pi(1 - e^{-\gamma\pi})}$$

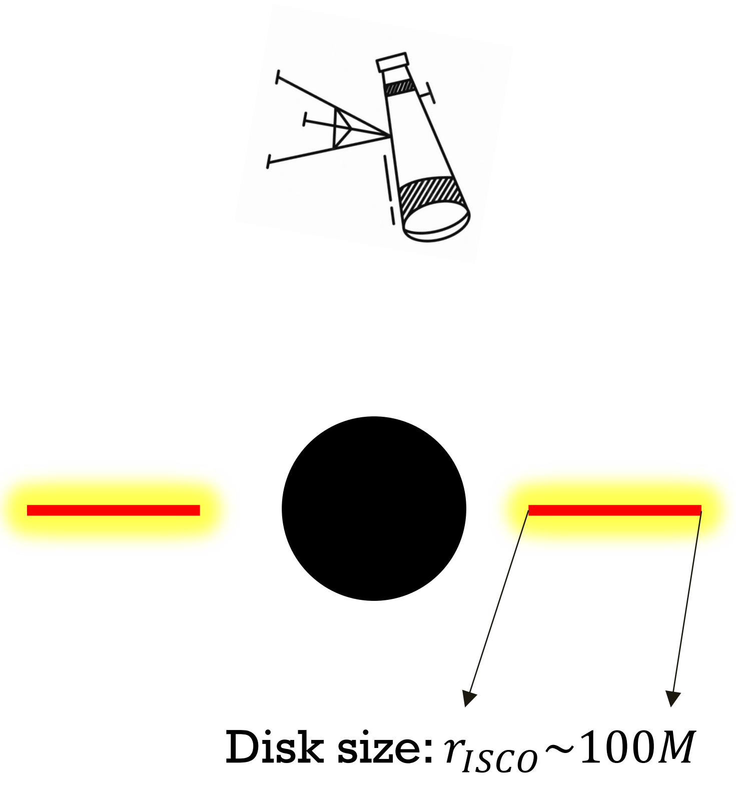
- We find $C > 0$ for most cases of interest
- This implies $\epsilon_n < \epsilon_{n+1} < 0$
- Two ring observables converge to γ from above when $n \to \infty$



Chen *et al.* (2022)

Example: Reissner-Nordström Black Holes





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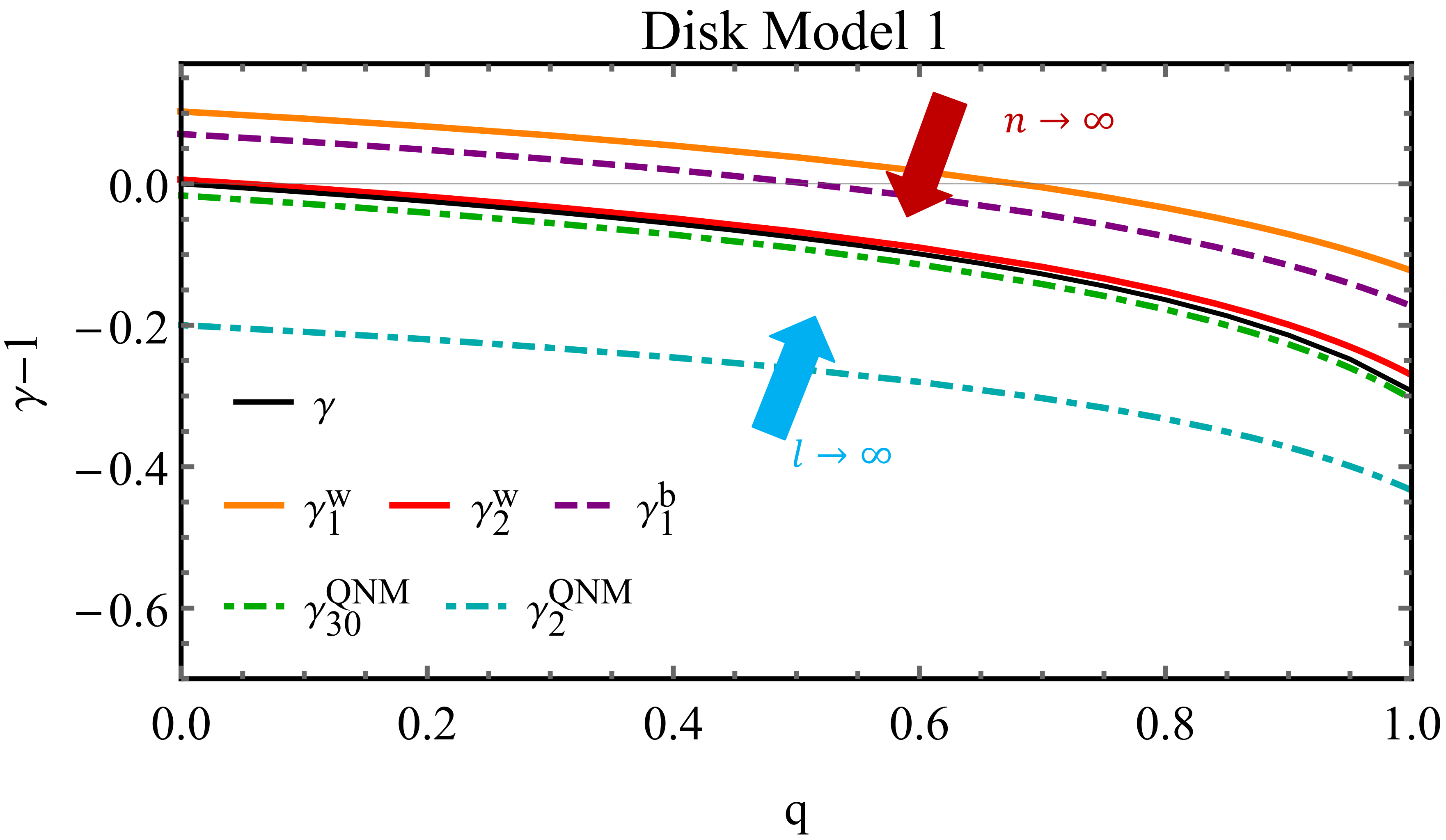
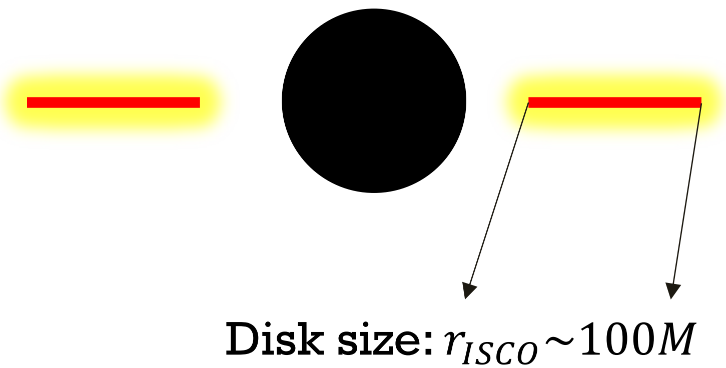
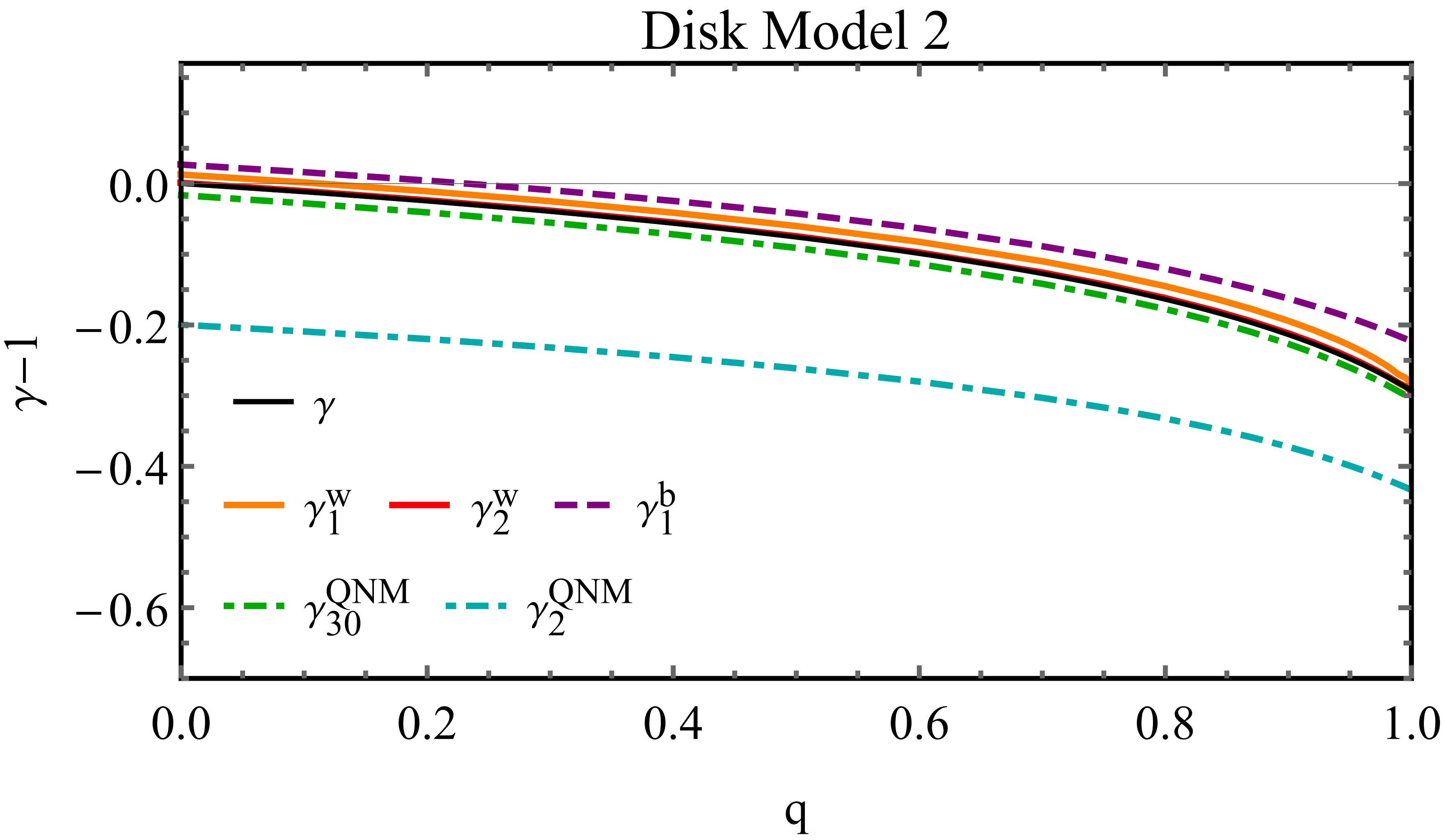
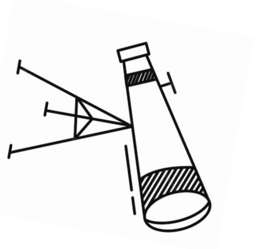
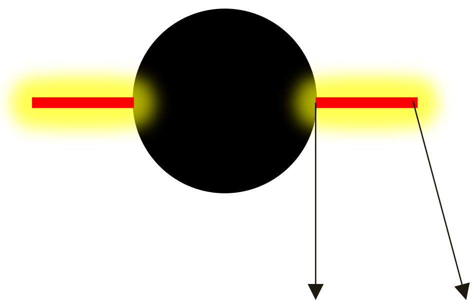




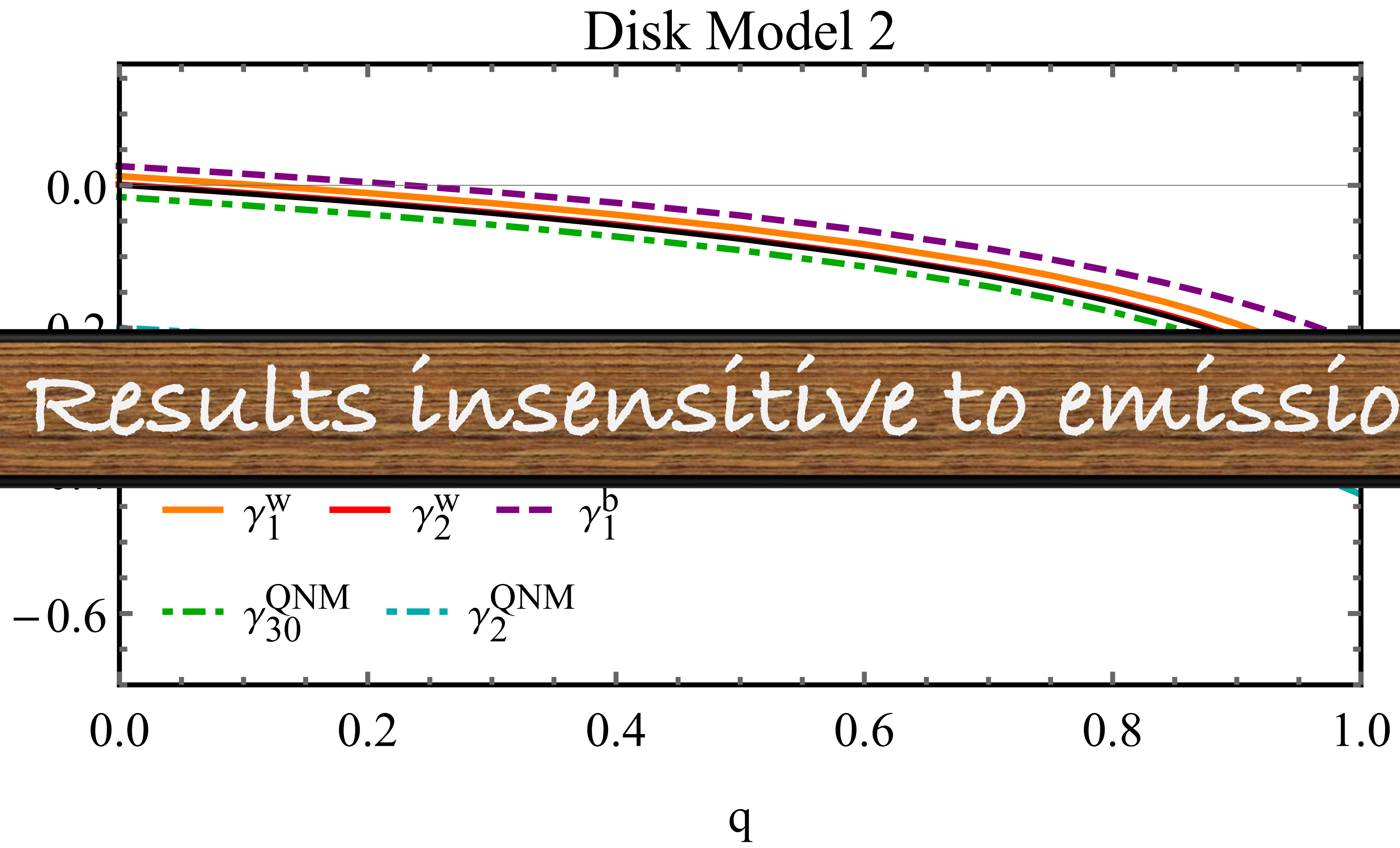
Example: Reissner-Nordström Black Holes

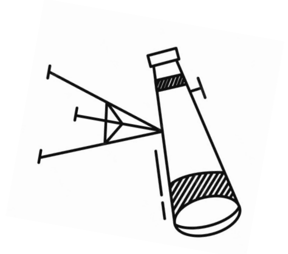


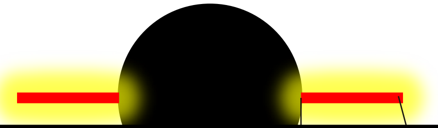




Example: Reissner-Nordström Black Holes



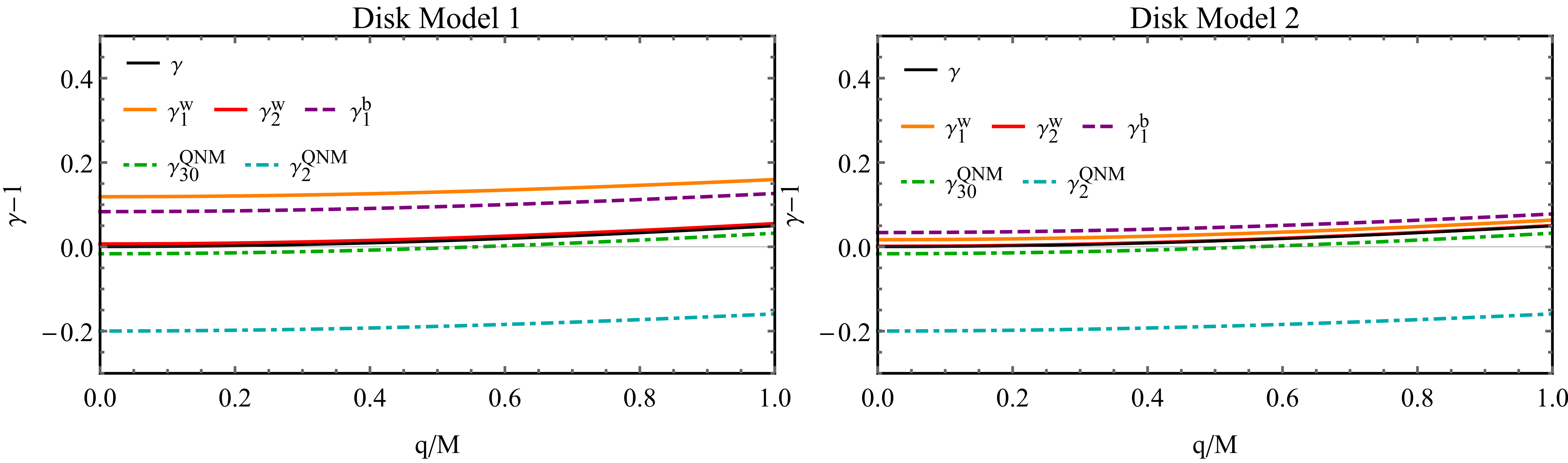




Disk size: $r_{EH} \sim 10M$

Results insensitive to emission models!

Example: Kazakov-Solodukhin Black Holes



- Robust qualitative converging tendency in different metrics

Example: Dynamical Chern-Simons Gravity

$$S = \int d^4x\sqrt{-g}\left(\kappa R + \frac{\alpha}{4}\vartheta R R^*\right) - \frac{\beta}{2}\int d^4x\sqrt{-g}(\partial\vartheta)^2$$

- Parity-violating term from the CS correction

Jackiw, Pi (2003) Alexander, Yunes (2009)

- Motivated from string theory

Campbell, Kaloper, Madden, Olive (1993) Moura, Schiappa (2006)

- Schwarzschild metric: an exact vacuum solution
- Schwarzschild perturbations: Axial mode coupled to scalar modes

Cardoso, Gualtieri (2010) Molina, Pani, Cardoso, Gualtieri (2010) Motohashi, Suyama (2011)(2012) Kimura (2018)

- The modes violate eikonal correspondence

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CS correction

dynamical scalar field

coupled QNM equation: $$\left(\frac{d^2}{dr_*^2} + \omega^2\right)\begin{pmatrix}\Psi \\ \Theta\end{pmatrix} = \begin{pmatrix}V_{11} & V_{12} \\ V_{21} & V_{22}\end{pmatrix}\begin{pmatrix}\Psi \\ \Theta\end{pmatrix}$$

Ψ : axial mode

Θ : scalar mode

$$V_{11} = \left(1 - \frac{2M}{r}\right)\left(\frac{l(l+1)}{r^2} - \frac{6M}{r^3}\right), \quad V_{22} = \left(1 - \frac{2M}{r}\right)\left(\frac{l(l+1)}{r^2}\left(1 + \frac{36M^2}{\kappa\beta r^6}\right) + \frac{2M}{r^3}\right)$$

$$V_{12} = V_{21} = \left(1 - \frac{2M}{r}\right)\sqrt{\frac{(l+1)!}{\beta\kappa(l-1)!}}\frac{6M}{r^5}$$

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CS correction   dynamical scalar field

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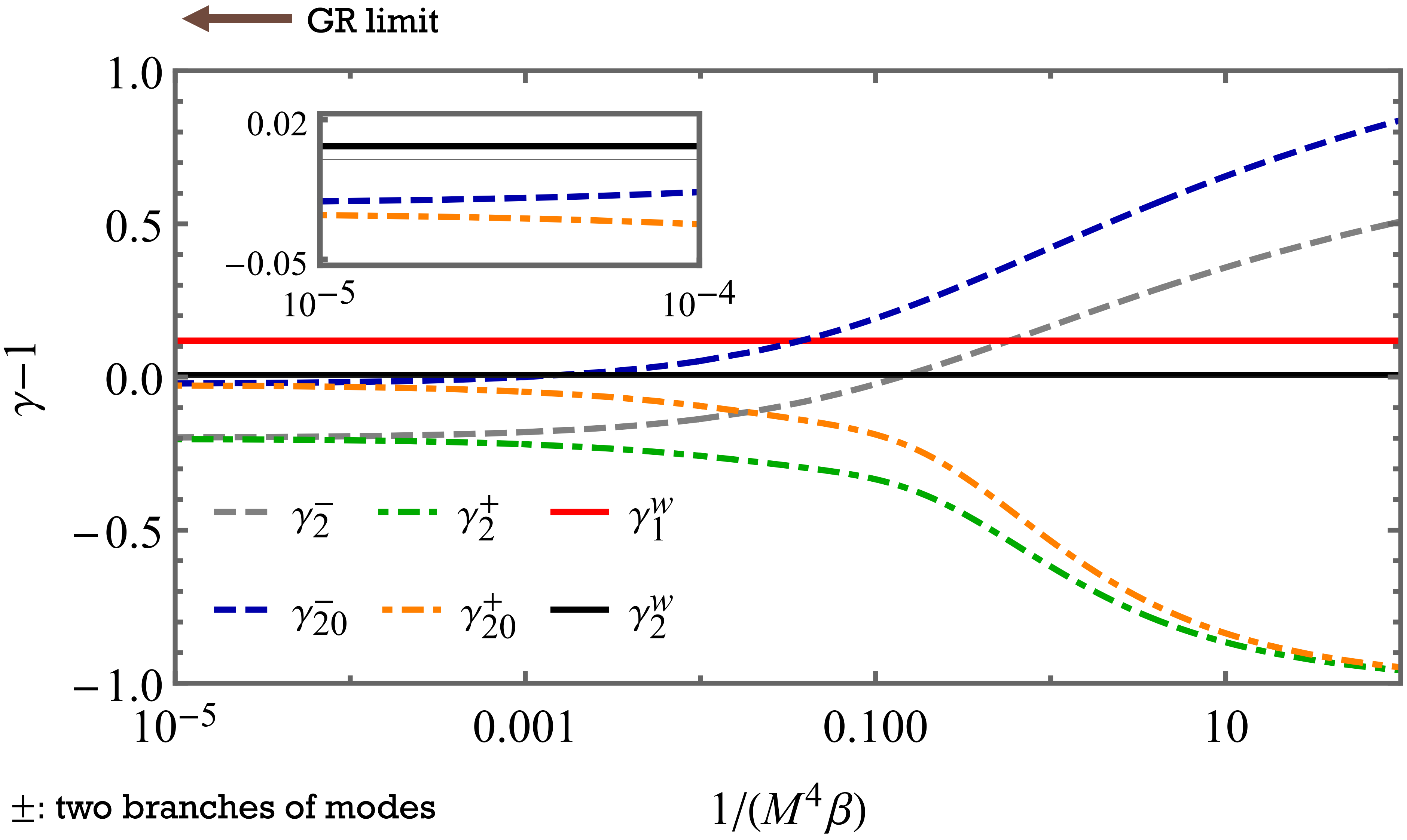
break eikonal correspondence

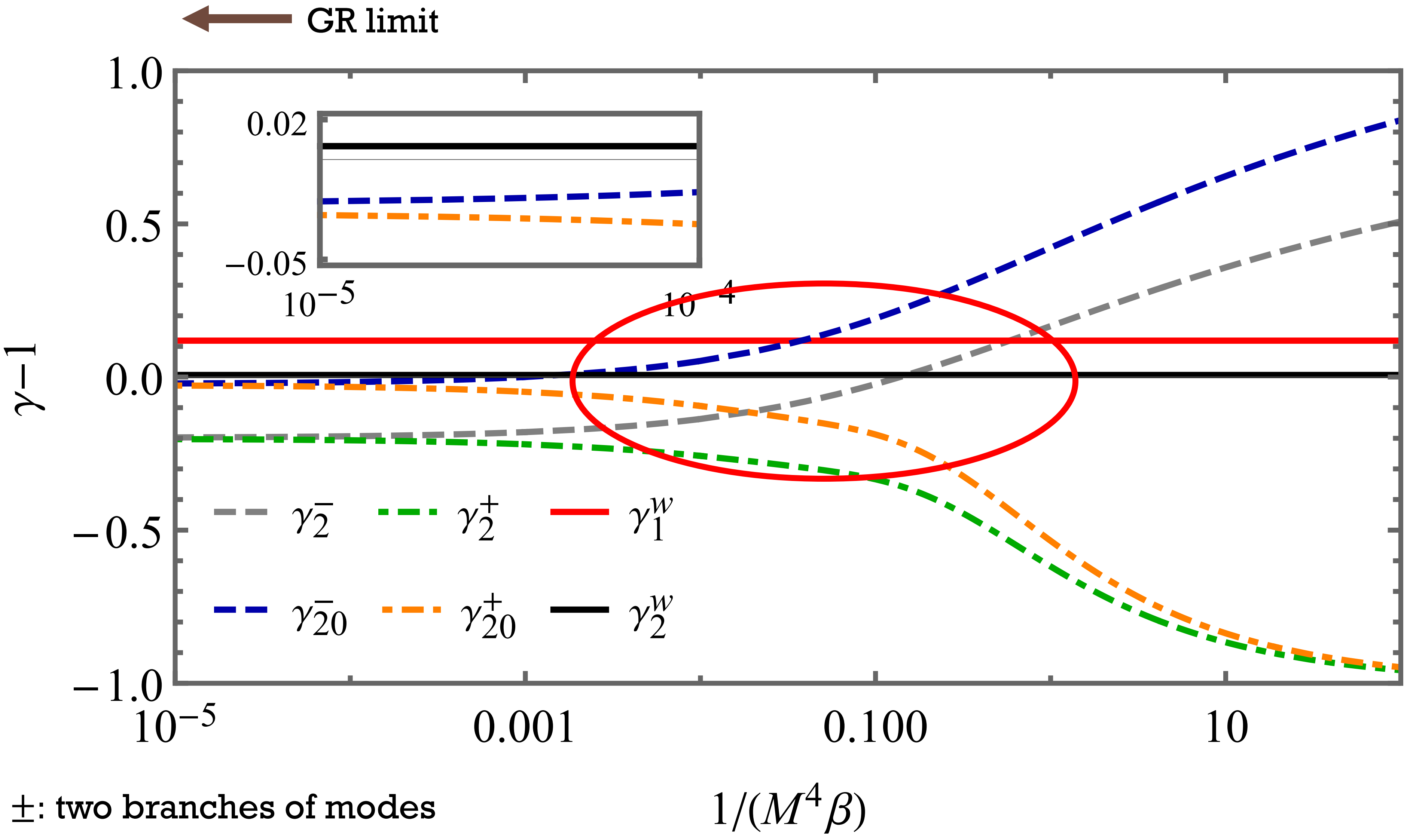
$$V_{12} = V_{21} = \left(1 - \frac{2M}{r}\right)\sqrt{\frac{(l+1)!}{\beta\kappa(l-1)!}}\frac{6M}{r^5}$$

Example: Dynamical Chern-Simons Gravity



±: two branches of modes

Example: Dynamical Chern-Simons Gravity



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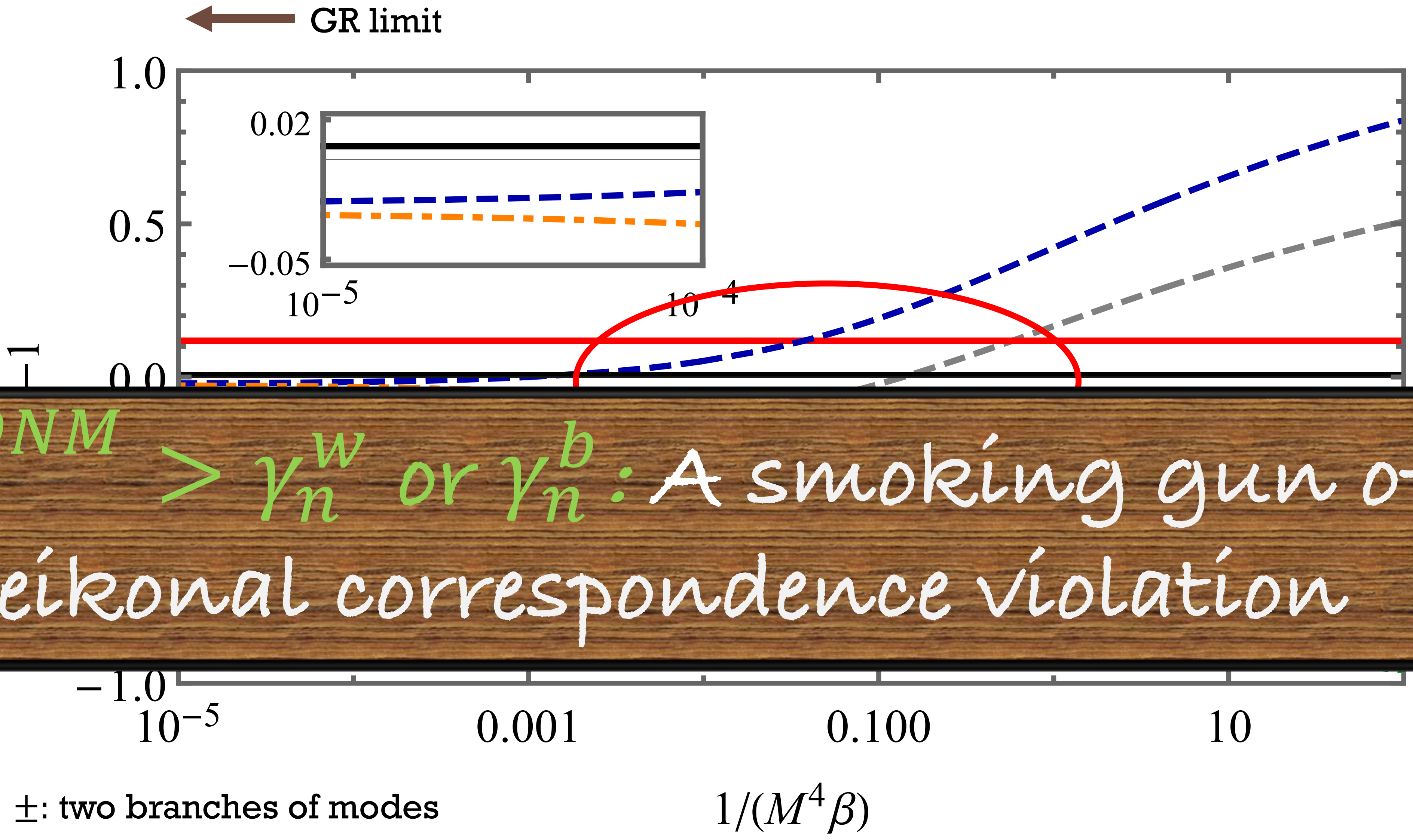
Example: Dynamical Chern-Simons Gravity



$\gamma_l^{QNM} > \gamma_n^w$ or γ_n^b: A smoking gun of eikonal correspondence violation

±: two branches of modes

- Black hole QNMs & WKB method

- Eikonal correspondence between QNMs and shadows
  - Basis
  - Testing gravity
- Conclusions

Conclusions

- WKB method
  - semi-analytic method
  - Powerful for calculating QNMs with $l \gtrsim n$
- Eikonal correspondence between QNMs and shadows
- Testing the correspondence
  - Testing GR through joint observations of GW and images