# For this afternoon:
## … if you're participating

- Please download GABE
  - cosmo.kenyon.edu/gabe.html
- If you have a MAC
  - Try to run the instructions on that page, new(ish) operating systems should work without issue
- If you have LINUX
  - Please install fftw3 (enabling regular and long-double: openmp and threads for each)
- If you plan to use the remote version, please email me (giblinj@kenyon.edu) as soon as possible!
- **Please also have Jupyter (or similar) for plotting. I will distribute a .ipynb this afternoon!**

# Numerical Preheating

## Thoughts and Exercises: Morning

**Tom Giblin**
**RIKEN Interdisciplinary Theoretical and Mathematical Sciences Program**
**Saitama, Japan**
**March 5, 2025**

# My goal

- I’m not going to try to sell you anything (although if you want to buy it…)
- There are some (beautiful) subtleties associated with doing nonlinear dynamics
  - This will not be an exhaustive list
- **Please talk with us (all of the people who do numerical work)**
  - We’re (always) happy to share ideas on figuring out whether results are *physical* or *numerical*

# Simulating Preheating

## Has a long history

- Perhaps the first real simulations are from 1996/7
  - S. Yu. Khlebnikov and Igor Tkachev
  - Phys. Rev. Lett. 77, 219 (1996)
- Made ‘famous’ by LatticeEasy around 2000
  - Gary Felder and Igor Tkachev
  - Comput.Phys.Commun.178:929-932.2008
- LatticeEasy opened the door to (any of us) to look at nonlinear dynamics of *any* inflationary model

# There are many (many) codes

## …many of which are open-source or available

- LatticeEasy, Gary Elder 2000 (https://www.felderbooks.com/latticeeasy/index.html)
    - CLUSTEREasy, arXiv:0712.0813 (https://www.felderbooks.com/latticeeasy/index.html)
- DEFrost, Andrei Frolov, 2008, arXiv:0809.4904, (https://www.sfu.ca/physics/cosmology/defrost/)
- CUDAEasy, Jani Sanio, 2009, arXiv:0911.5692
- PSpectRe, Richard Easther, Hal Finkle, Nathaniel Roth, arXiv:1005.1921
- HLATTICE, Zhiqi Huang, arXiv: 1102.0227 (https://www.cita.utoronto.ca/~zqhuang/hlat/)
- GABE, JTG, Hillary Child, J. Tate Deskins, arXiv:1305.0561, (https://cosmo.kenyon.edu/gabe.html)
- PyCOOL, Jani Sainio, arXiv:1201.5029
- CosmoLattice, 2020, Daniel G. Figueroa, Adrien Florio, Francisco Torrenti, Wessel Valkenburg, arXiv:2006.15122, (https://cosmolattice.net/)

# AND I'm missing some

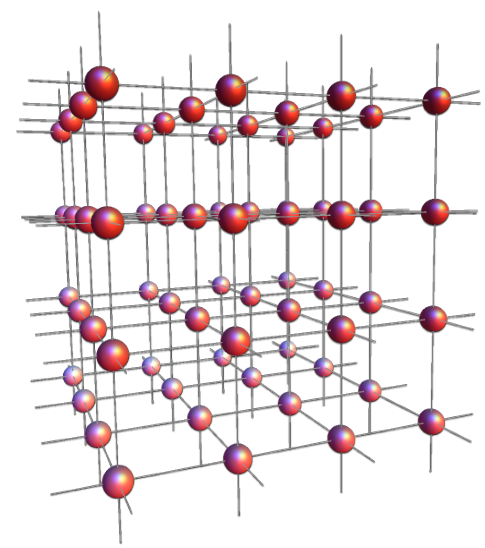
- This list doesn't include programs
  - (like CACTUS) that were designed for simulating scalar fields in other contexts
  - That were written for Numerical Relativity and can handle scalar fields
  - Etc, etc
  - I apologize in advance for any citations or contributions that I've left off!!

# What’s the primary take-away?

# The discrete system *is* a physical system

## …but it's not the same physical system as the continuum

$\phi(\vec{x})$



$\phi(\vec{x}_i)$

# The simplest (and most relevant) example

## …the ideal numerical system*

- Consider the wave equation: $\ddot{\phi} - \nabla^2 \phi = 0$

- Which is the coupled system of first-order PDEs:

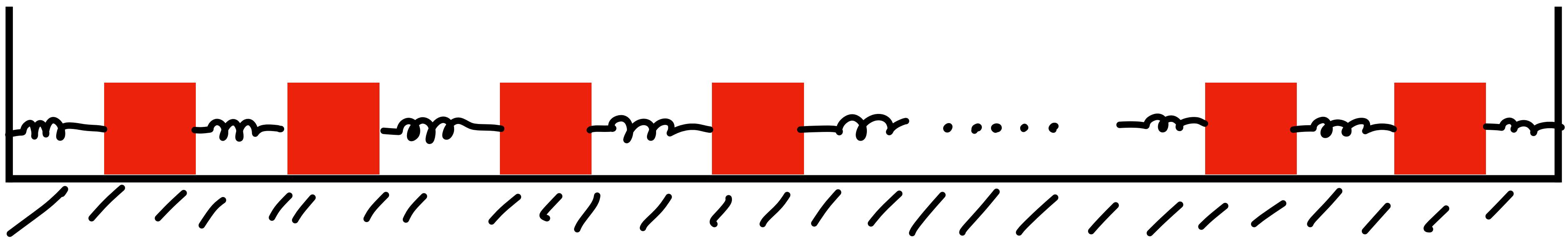
$$\dot{\phi} = \omega$$
$$\dot{\omega} = \nabla^2 \phi$$

- Which are, in the discrete limit, a set of $N$ (or $N^3$) coupled ODE's**:

$$\dot{\phi}(\vec{x}_i) = \omega(\vec{x}_i) \qquad \dot{\omega}(\vec{x}_i) = \frac{1}{\Delta x^2}\left[\phi(\vec{x}_{i+1}) - 2\phi(\vec{x}_i) + \phi(\vec{x}_{i-1})\right]$$

*due to its strong hyperbolicity **written in 1-d

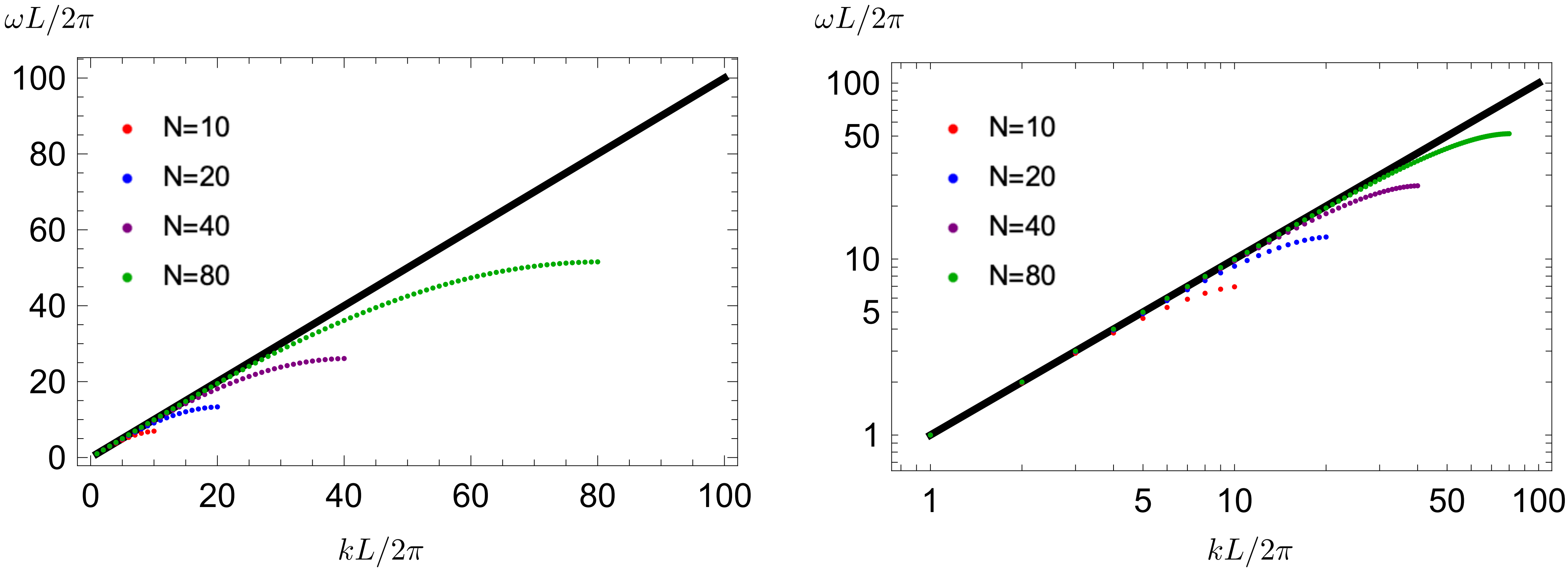
# Because you remember blocks and springs



- This system of discrete blocks approximates a continuum system
- However, the frequencies of the normal modes of the system come from find the eigenvalues of the system of equations

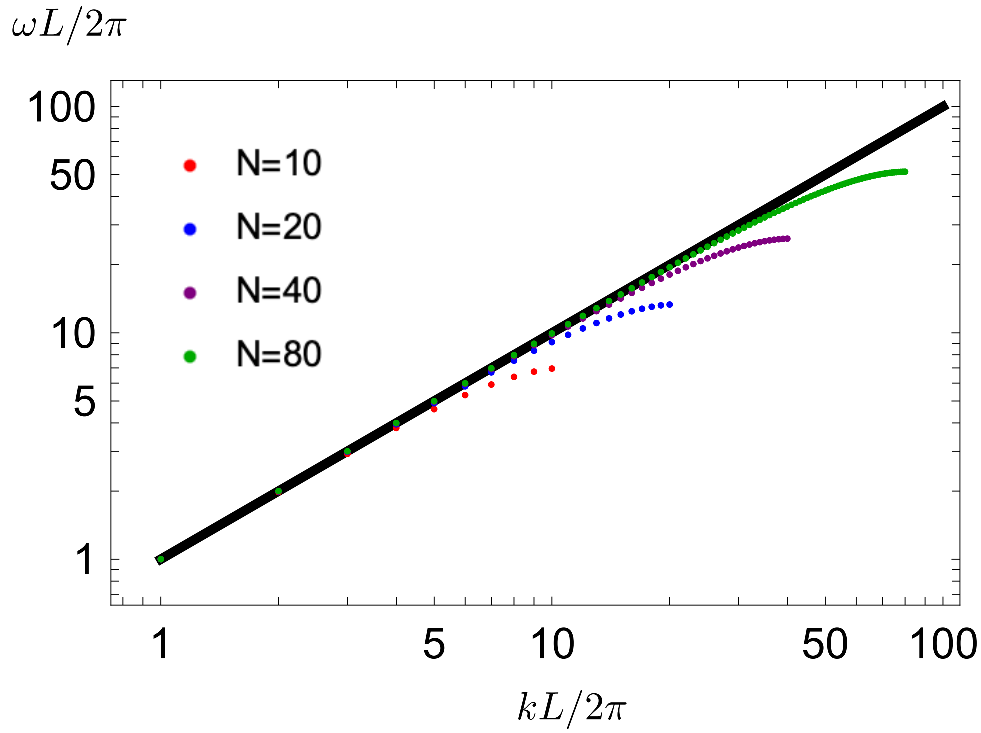
$$\ddot{\vec{y}} = \frac{k}{m} \begin{pmatrix} -2 & 1 & 0 & 0 & \\ 1 & -2 & 1 & 0 & \\ 0 & 1 & -2 & 1 & \\ 0 & 0 & 1 & -2 & \\ & & & & \ddots \end{pmatrix} \vec{y}$$

# The dispersion relation for this system



# The dispersion relation for this system

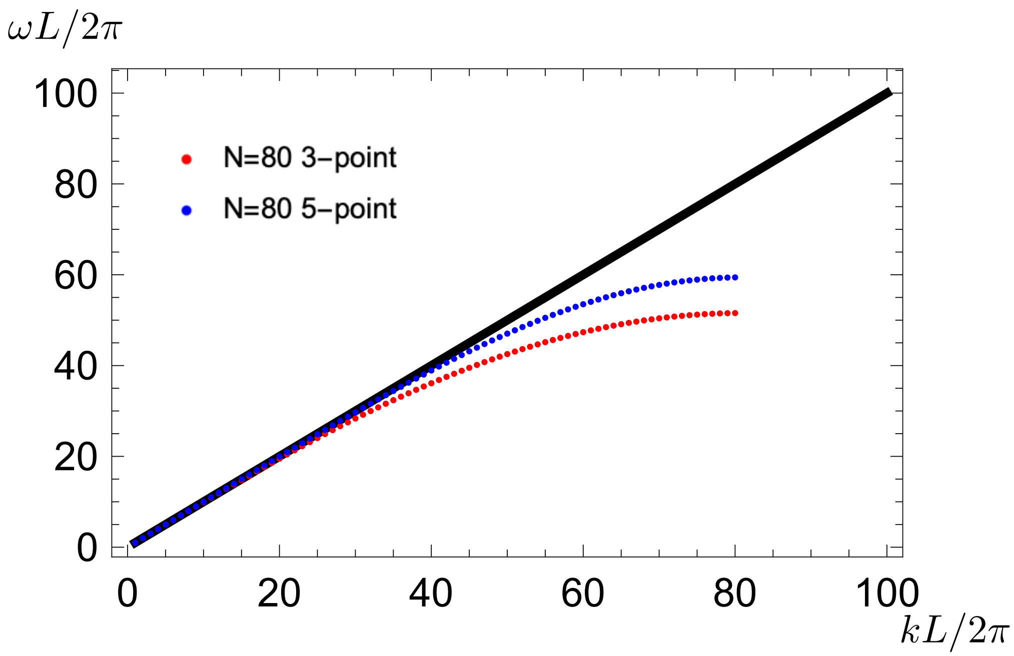
- Remember that the more *points you have, the larger the wavenumber you can resolve!*
- The box-size sets the *minimum* (non-zero) wavenumber that you can resolve
    - This also means that all wavelengths larger than this are “included” in the zero-bin

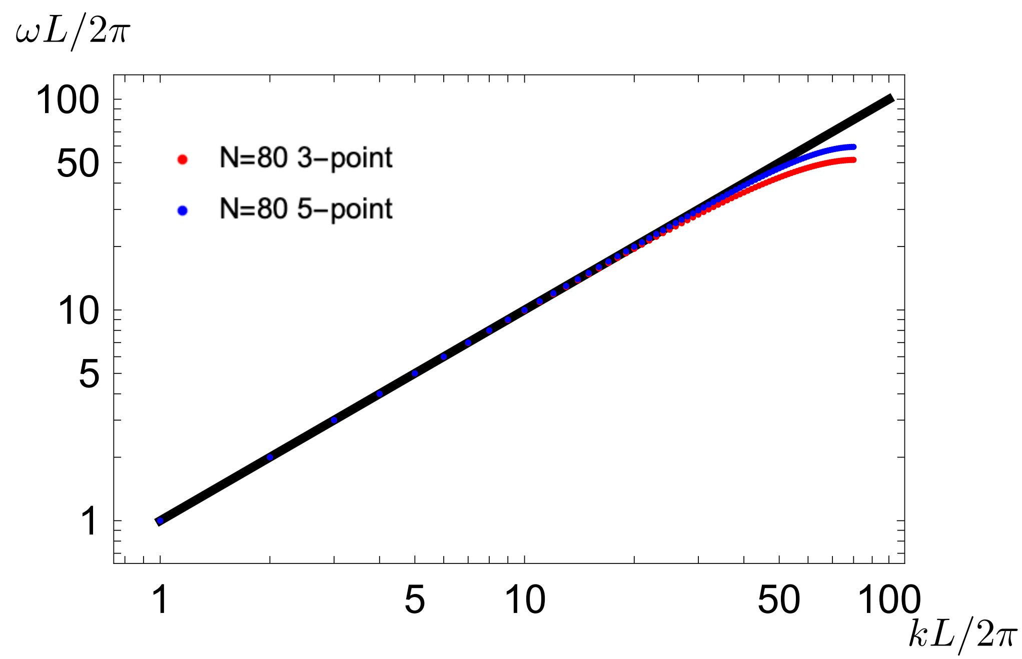


# You can play some 'tricks'

## ...like stencils

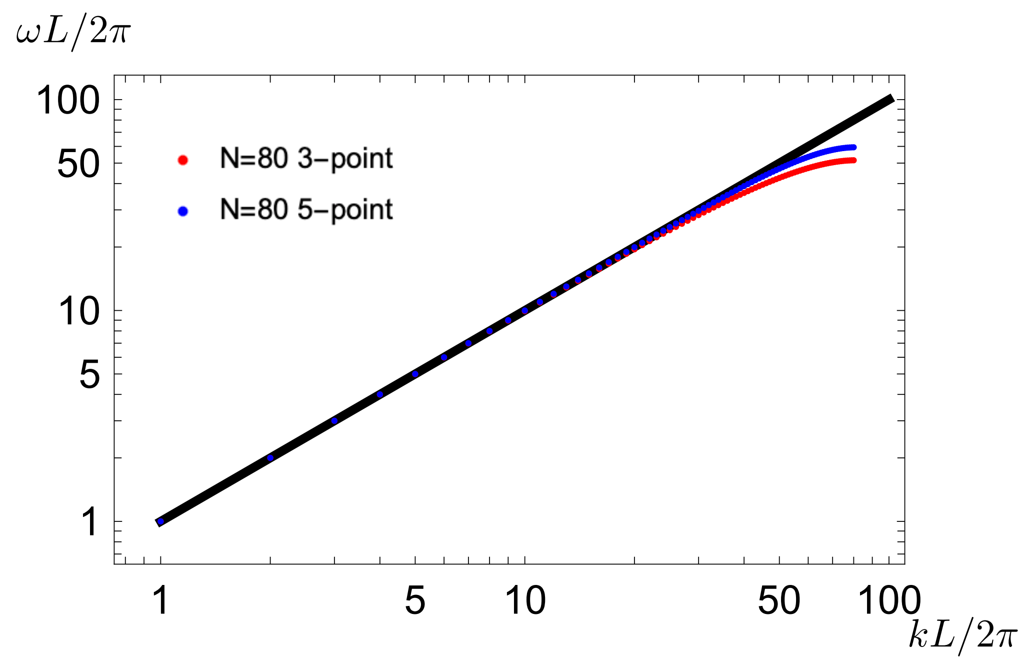
$$\dot{\omega}(\vec{x}_i) = \frac{1}{12\Delta x^2}\left[-\phi(\vec{x}_{i-2}) + 16\phi(\vec{x}_{i+1}) - 30\phi(\vec{x}_i) + 16\phi(\vec{x}_{i-1}) - \phi(\vec{x}_i)\right]$$



# You have to look where you can *trust*

- To quantify how much you "trust" you need to do more sophisticated tests
    - More on these later
- But in general, you plan to have the physics you care about in the lower-half of the log-modes



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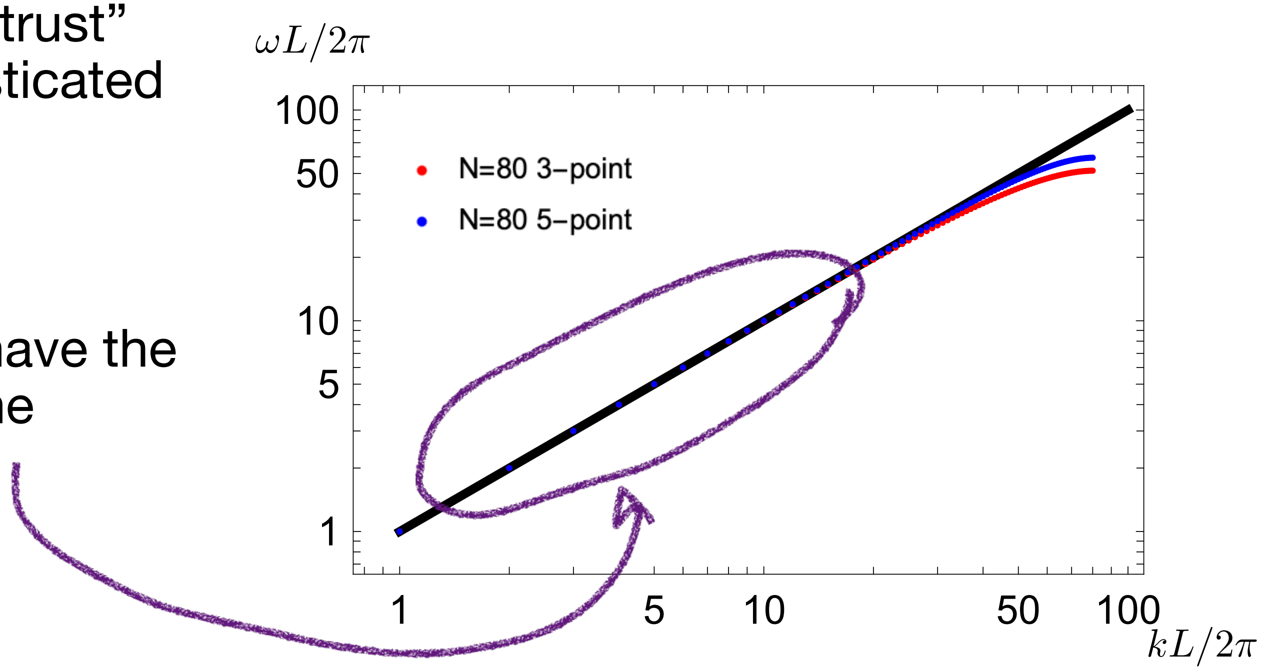


# A little bit about GABE

# Grid and Bubble Evolver (GABE)

- A discritized lattice of $N^3$ points
- $2^{\text{nd}}$-order Runge-Kutta Integration scheme
- Natively it handles $n$-scalar fields
  - Which are treated completely non-linearly
- Gravity is treated homogeneously via the Friedman constraint
- It's "easy" to change potentials
  - And *possible* to change equations of motion

The actual Gabriel

# Setting up the physical system

- Set the *scale(s)* of the problem by defining dimensionless variables
- Make choices for the *physical* parameters of the system
- Make choices for the *numerical* parameters of the system
    - Chose a scheme to discretize the system
    - Be aware that you will need to vary these parameters in order to validate your numerical work

# Defining dimensionless variables

## …the hard part

- The two (structural) parts of the system are:
  - Spacetime:

$$dx^\mu = \frac{dx^\mu_{\rm pr}}{B}$$

  - Gravity:

$$\left(\frac{\dot{a}}{a}\right)^2 = B^2 \left(\frac{a'}{a}\right)^2 = \frac{8\pi}{3m_{\rm pl}^2}\rho \qquad {}' \equiv \frac{\partial}{\partial t_{\rm pr}}$$

$$\rho_{\rm pr} = \frac{\rho}{m_{\rm pl}^2 B^2}$$

# Rescaling the fields

## …is also not a choice

- The fields are always in Planck masses,

$$\phi_{\rm pr} = \frac{\phi}{m_{\rm pl}}$$

- So *derivatives* of fields have a factor of $B$ as well as a factor of $m_{\rm pl}$, e.g.

$$\dot{\phi} = B m_{\rm pl} \frac{\partial \phi_{\rm pr}}{\partial t_{\rm pr}}$$

# For scalar fields only,

## Setting the scale(s) of the problem

- Rescaling the energy density is really just rescaling the potential (model)

$$\rho_{\rm pr} = \frac{\frac{1}{2}\dot{\phi}^2 + \frac{(\nabla\phi)^2}{2a^2} + V(\phi)}{B^2 m_{\rm pl}^2} = \frac{1}{2}{\phi'_{\rm pr}}^2 + \frac{(\nabla_{\rm pr}\phi_{\rm pr})^2}{2a^2} + \frac{V}{B^2 m_{\rm pl}^2}$$

- So the scale of the box is really set by the parameters of the potential, e.g.

$$V_{\rm pr} = \frac{V}{B^2 m_{\rm pl}^2} = \frac{1}{B^2 m_{\rm pl}^2}\frac{1}{2}m^2\phi^2 = \frac{1}{2}\phi_{\rm pr}^2 \qquad \text{where} \qquad B = m$$

# If the potential is more complicated

## Setting the scale(s) of the problem

- For example, Axion Monodromy,

$$V_{\rm pr} = \frac{m^2 M^2}{B^2 m_{\rm pr}^2}\left(\sqrt{1+\frac{\phi^2}{M^2}}-1\right)$$

- There appear to be many choices of $B$ that could simplify this potential; however, the choice $B = m$ leads to

$$V_{\rm pr} = \left(\frac{M}{m_{\rm pl}}\right)^2\left(\sqrt{1+\frac{\phi_{\rm pr}^2}{\left(M/m_{\rm pl}\right)^2}}-1\right) \approx \frac{1}{2}\phi_{\rm pr}^2 + \mathcal{O}\left(\phi_{\rm pr}^4\right)$$

# If the potential is more complicated

## Setting the scale(s) of the problem

- For example, Axion Monodromy,

$$V_{\rm pr} = \frac{m^2 M^2}{B^2 m_{\rm pr}^2}\left(\sqrt{1+\frac{\phi^2}{M^2}}-1\right)$$

- There appear to be many choices of $B$ that [obscured by overlay box] however, the choice $B=m$ leads to

Where the (small amplitude) oscillation of the homogeneous field still sets the clock

$$V_{\rm pr} = \left(\frac{M}{m_{\rm pl}}\right)^2\left(\sqrt{1+\frac{\phi_{\rm pr}^2}{(M/m_{\rm pl})^2}}-1\right) \approx \frac{1}{2}\phi_{\rm pr}^2 + \mathcal{O}\left(\phi_{\rm pr}^4\right)$$

# You only get one

## Setting the scale(s) of the problem

- However, the choice of $B$ "uses up" the freedom to set other scales of the problem.
    - In broad strokes, the number of parameters (beyond one) that you need to specify are the the root of the numerical challenges
- Other codes (most notably LatticeEasy) have more freedom in choosing dimensionless variables

$$\phi_{\rm pr} = A a^r \phi \qquad d\vec{x}_{\rm pr} = B\, d\vec{x} \qquad dt_{\rm pr} = B a^s dt \qquad s = 2r - 3$$

- However, nothing is “free” — these choices change couplings to gravity, e.g., which just shift around where you’re making your choices!

# The physical parameters
## Choosing the physics

- This could mean choosing masses, couplings, etc,

$$V = \frac{1}{2}m^2\phi^2 + \frac{g^2}{2}\phi^2\chi^2 \qquad m = 10^{-6}\,m_{\rm pl} \qquad g^2 = 2.5\times 10^{-7}$$

- Or the initial conditions (say, at the end of inflation)

$$\phi_0 = 0.193\,m_{\rm pl} \qquad \dot\phi_0 = -0.142\,m\,m_{\rm pl}$$

$$\phi_0^{\rm pr} = 0.193 \qquad \dot\phi_0^{\rm pr} = -0.142$$

- Which can also give other *physical* quantities of interest, e.g. $H$

# The physical choices *inform* the numerical

## …but they do not define them

- For vanilla preheating, we find

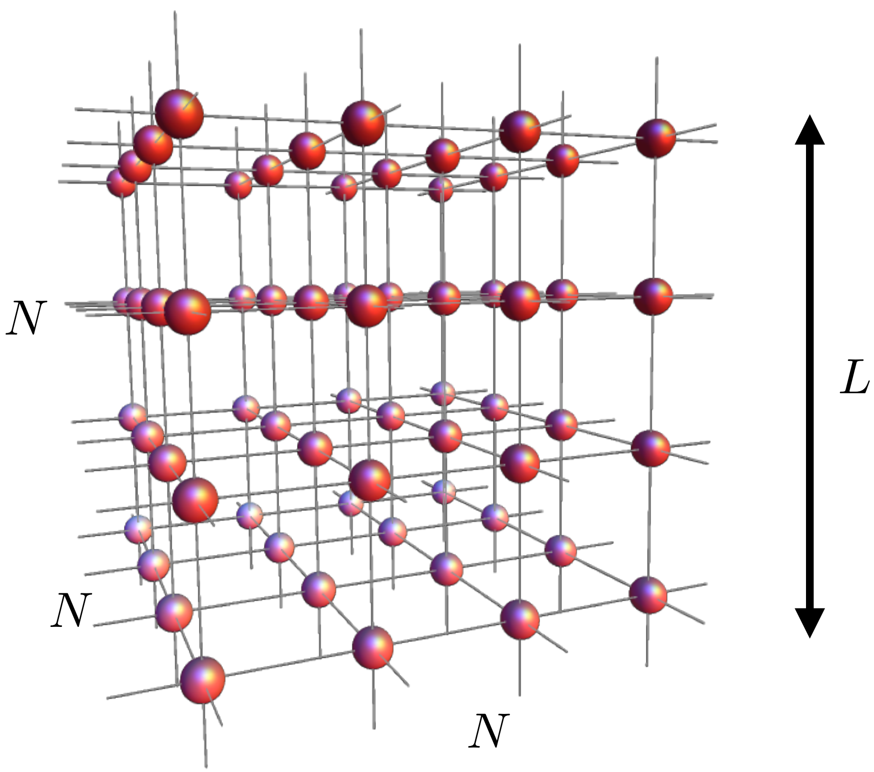
$$H_{\rm pr} \approx 0.49 \sim 0.5$$

- Which means that we *should* think about

$$L_{\rm pr} \sim 2$$

- BUT: this only suggests that the physics we're interested in are on *this scale*, and that choices of $L_{\rm pr} = 10$ or $L_{\rm pr} = 0.5$ should give similar phenomenology

# Basic Numerical Parameters




$$k_{\text{min}} = \frac{2\pi}{L} \longleftrightarrow \lambda_{\text{max}} = L$$

$$k_{\text{max}} = \frac{\sqrt{3}N}{2}\frac{2\pi}{L} \longleftrightarrow \lambda_{\text{min}} = \frac{L}{N}\frac{2}{\sqrt{3}} \approx \Delta x$$

# So what should L be?

## …what it needs to be

- You have to *vary* $L$ and $N$ (among others) to ensure that the *physics* you are looking for is independent of these choices.
- This is known as a *convergence test* — that the *physics* converges as the simulation more closely approximates the continuum
- This step is **crucial** to convince us that the simulations are predicting outcomes from the continuum theory.
  - **Regardless of any other measure (e.g. energy conservation), you must show convergence if we are to believe that the discrete system approximates the continuum.**

# And of course, $\Delta t$

- The Courant–Friedrichs–Lewy (or just Courant) Condition give *guidance* as to how small your timestep should be

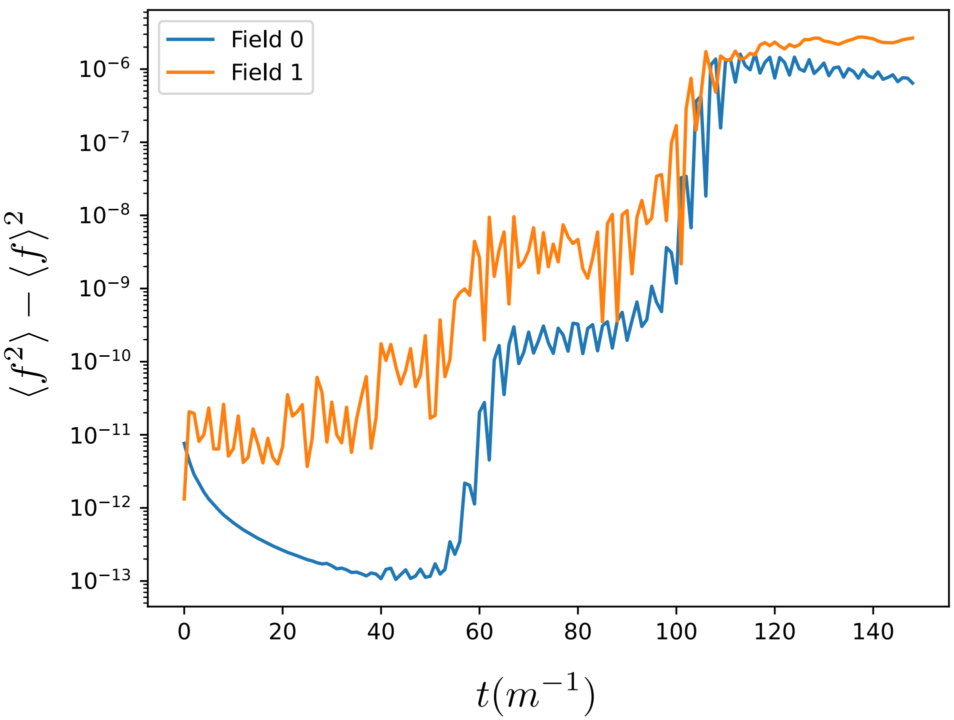
- Basically, you need enough *time* resolution to resolve the fastest-oscillating mode, e.g.

$$dt \lesssim T \lesssim \frac{2\pi}{\omega_{\max}} = 2\pi \frac{2}{\sqrt{3}} \frac{1}{2\pi} \frac{L}{N} \; \mathcal{O}(1) \times \Delta x$$

- But, we probably knew that already.  This guidance is only a place to *start*; you still have to run a *timestep* convergence test to ensure that the timestep is small enough.

# Even when you win, it might only be temporary

## …consider vanilla preheating




- This is a plot of the variance, bumpiness, of the fields as a function of time for the model/parameters that come “shipped” with GABE
- This is the classic, ‘vanilla preheating’ model where we have the three stages of preheating
- So we can look at the modes (particle production) over the course of this run

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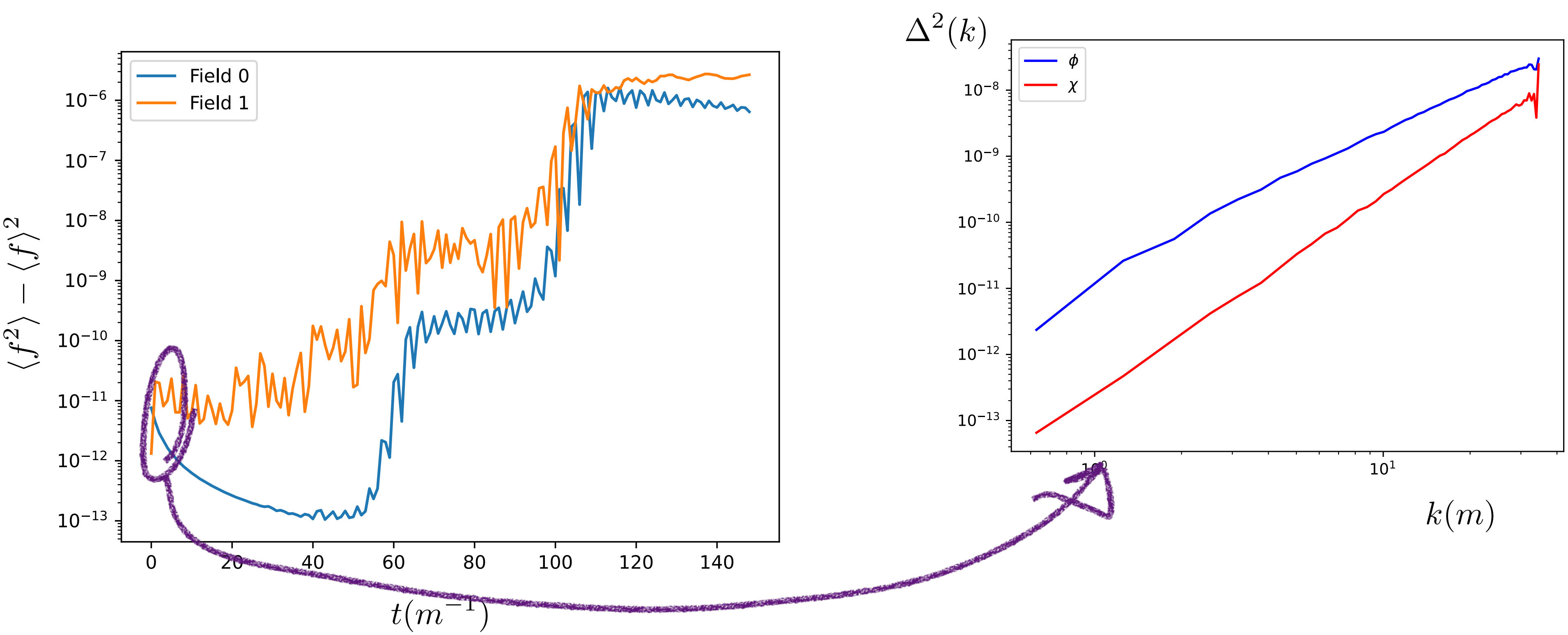

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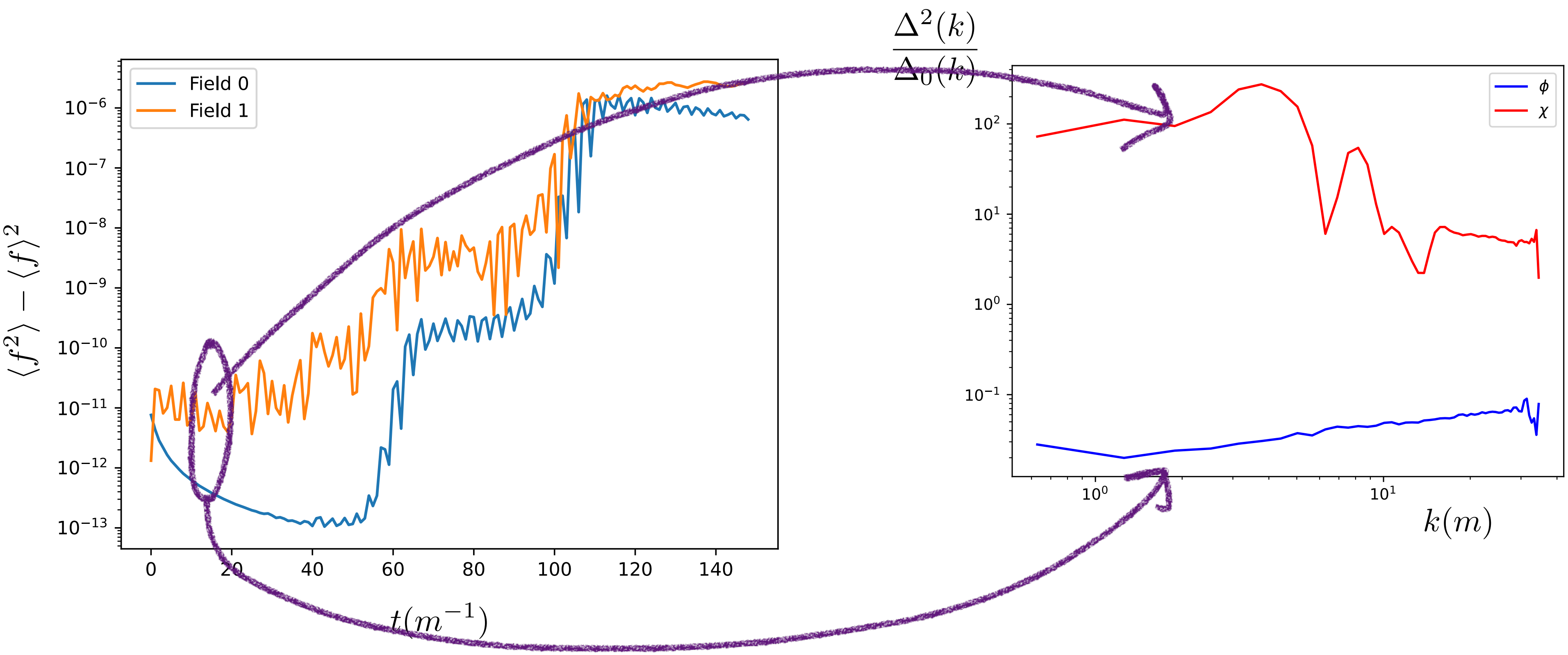

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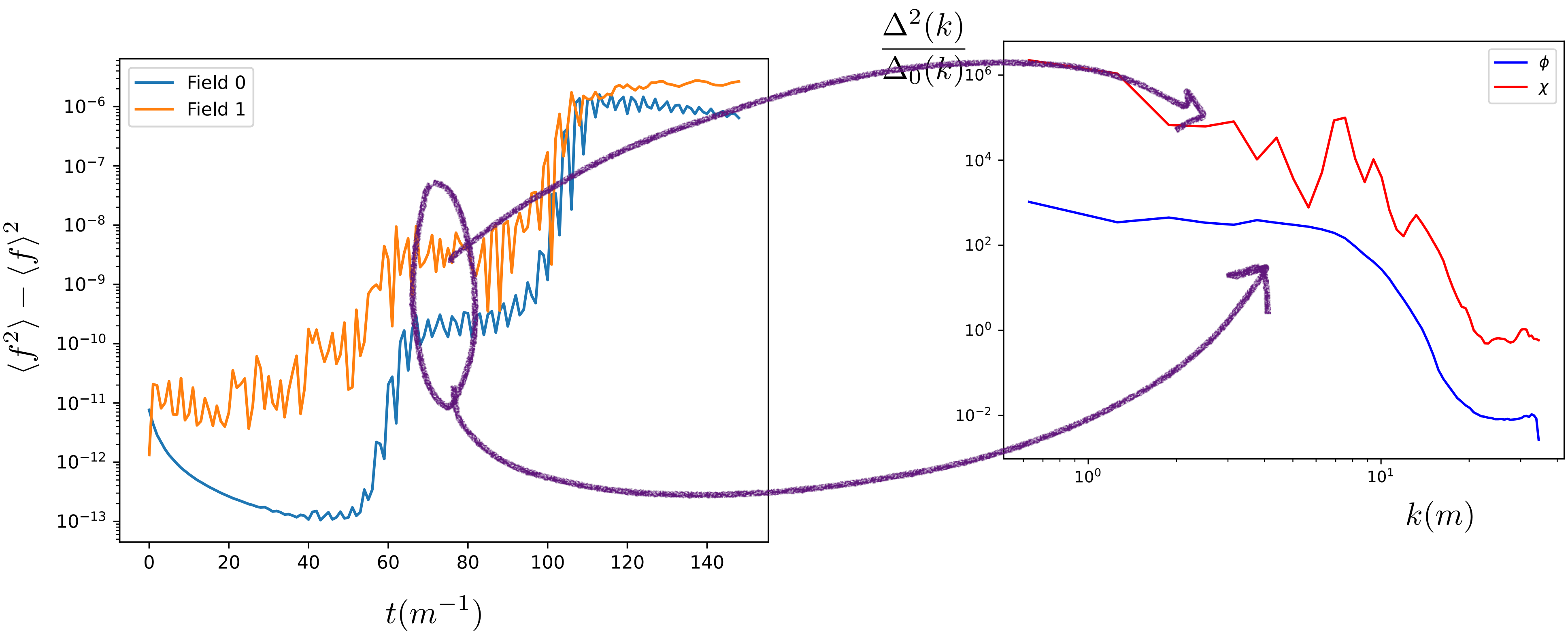

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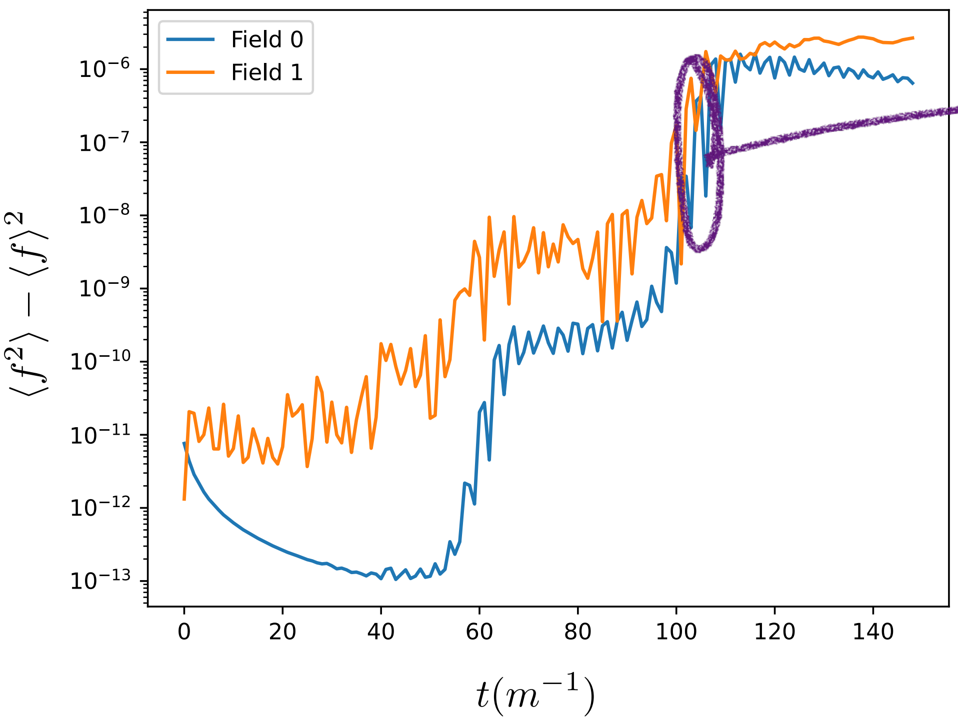


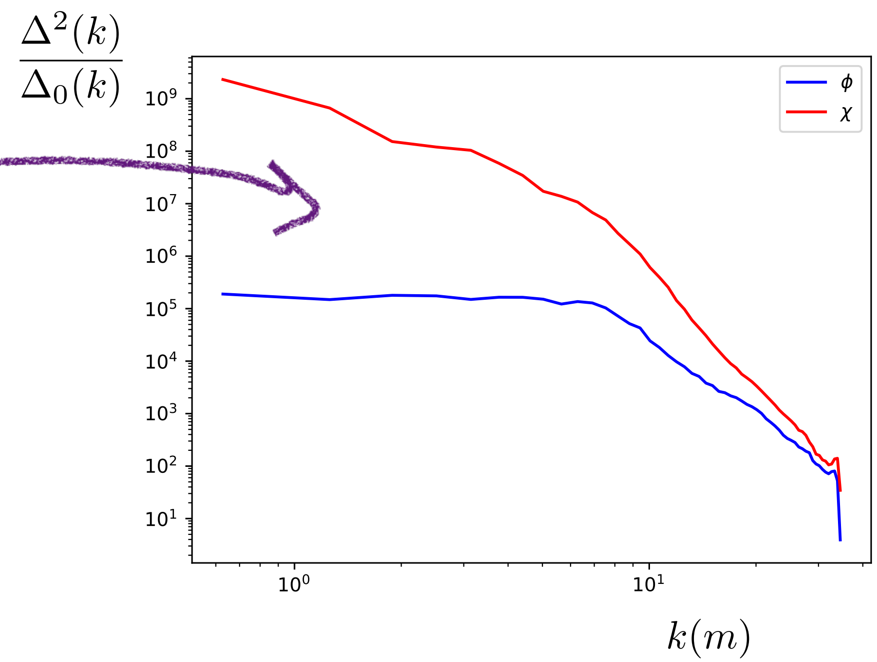

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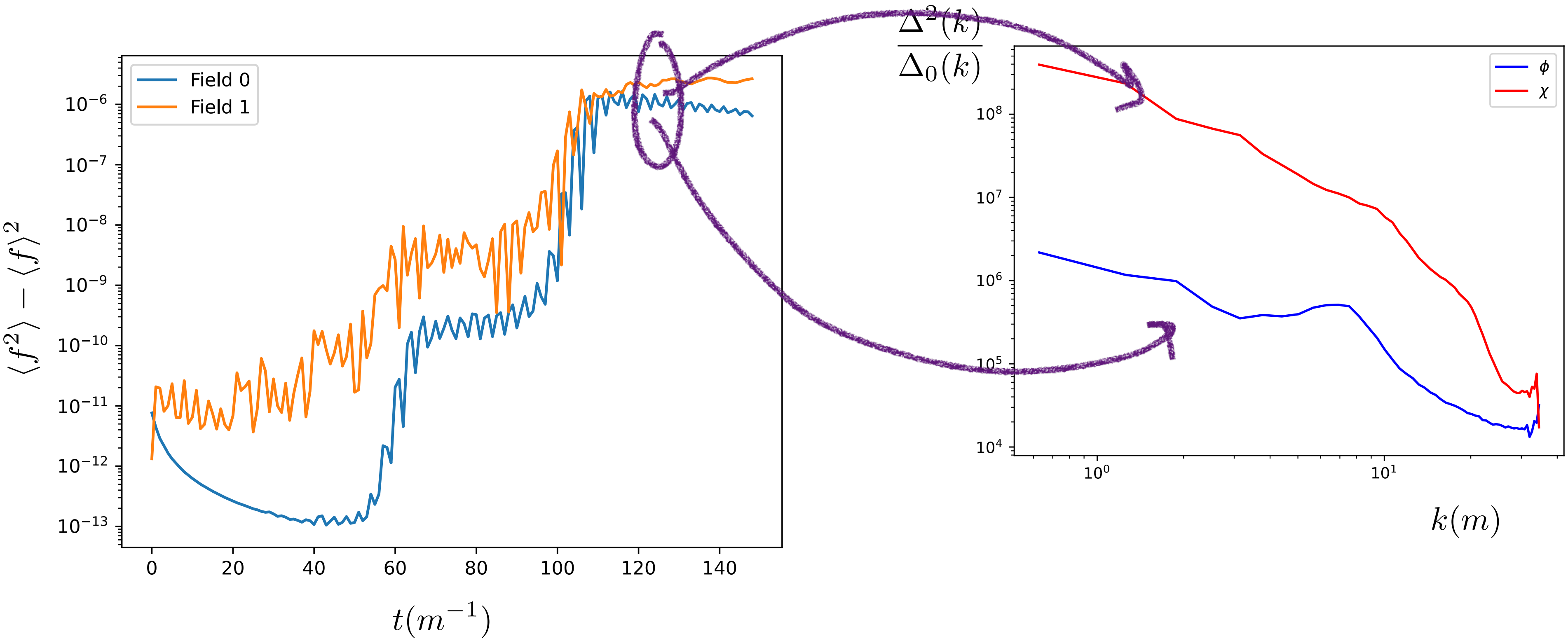


# Once power gets into those high modes

- Remember that gravitational waves

$$\ddot{h}_{ij} + 3H\dot{h}_{ij} - \nabla^2 h_{ij} = \frac{18\pi G}{a^2} S_{ij}^{\mathrm{TT}}$$

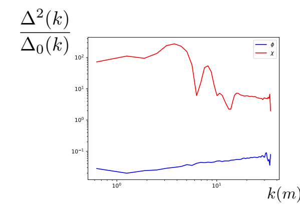
- Are sourced by non-linear combinations of derivatives,

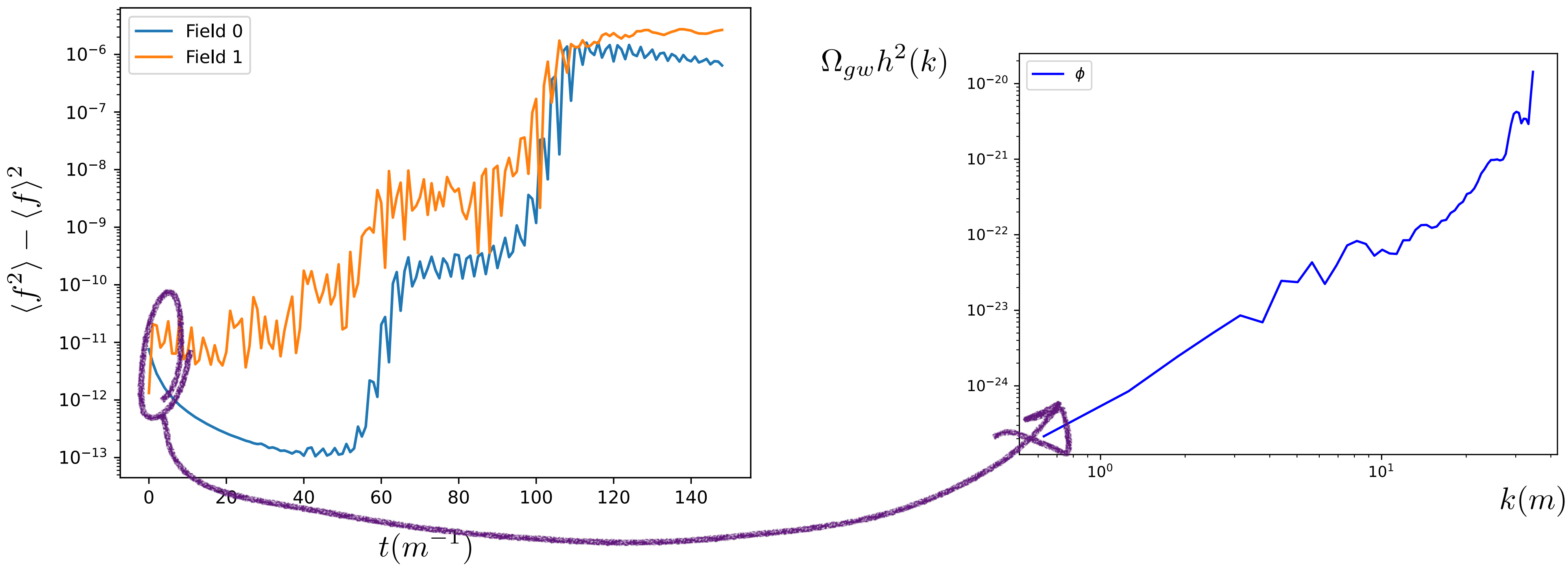
$$S_{ij} \sim \partial_i \phi \partial_j \phi$$

- They are very sensitive to any errors in high-frequency modes!

# Even when you win, it might only be temporary

## …consider vanilla preheating…with gravitational waves



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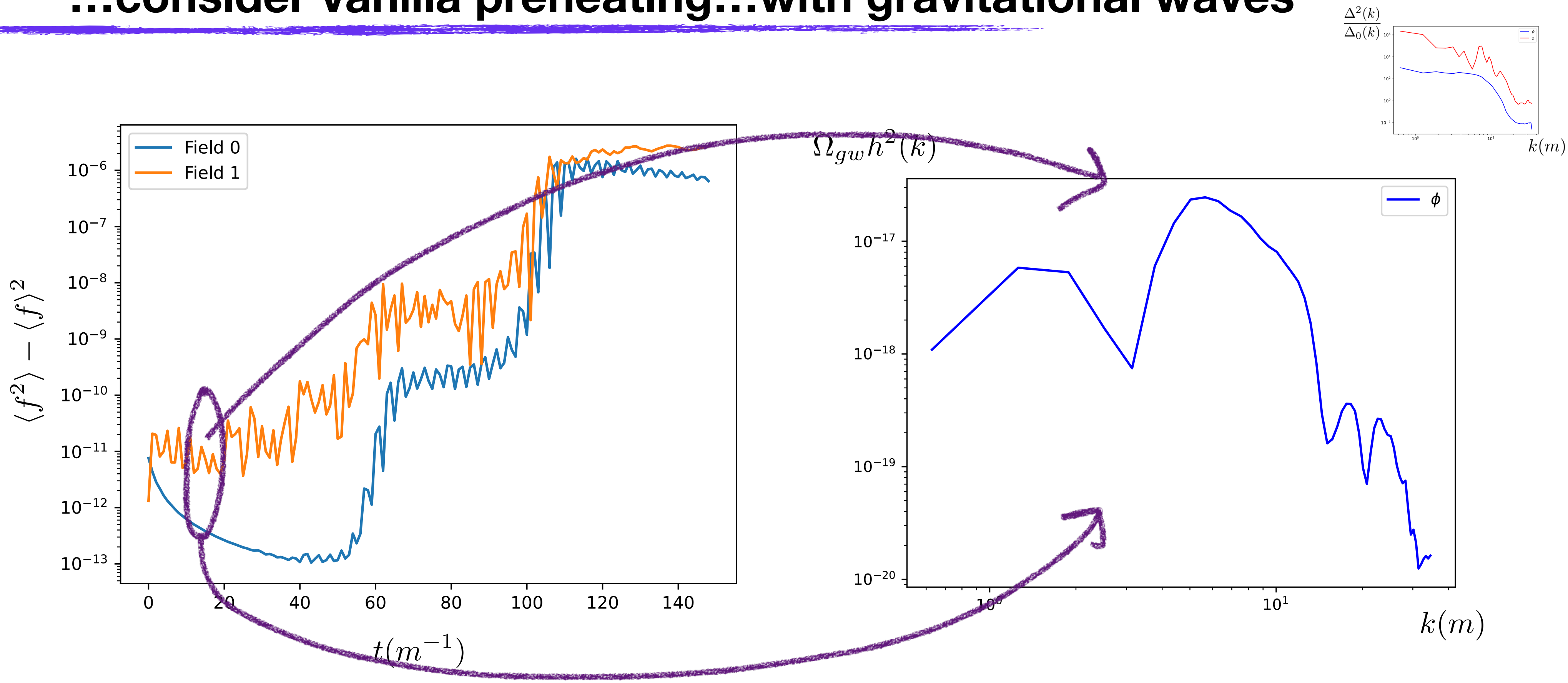

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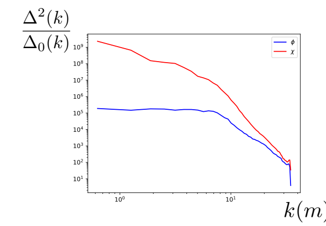


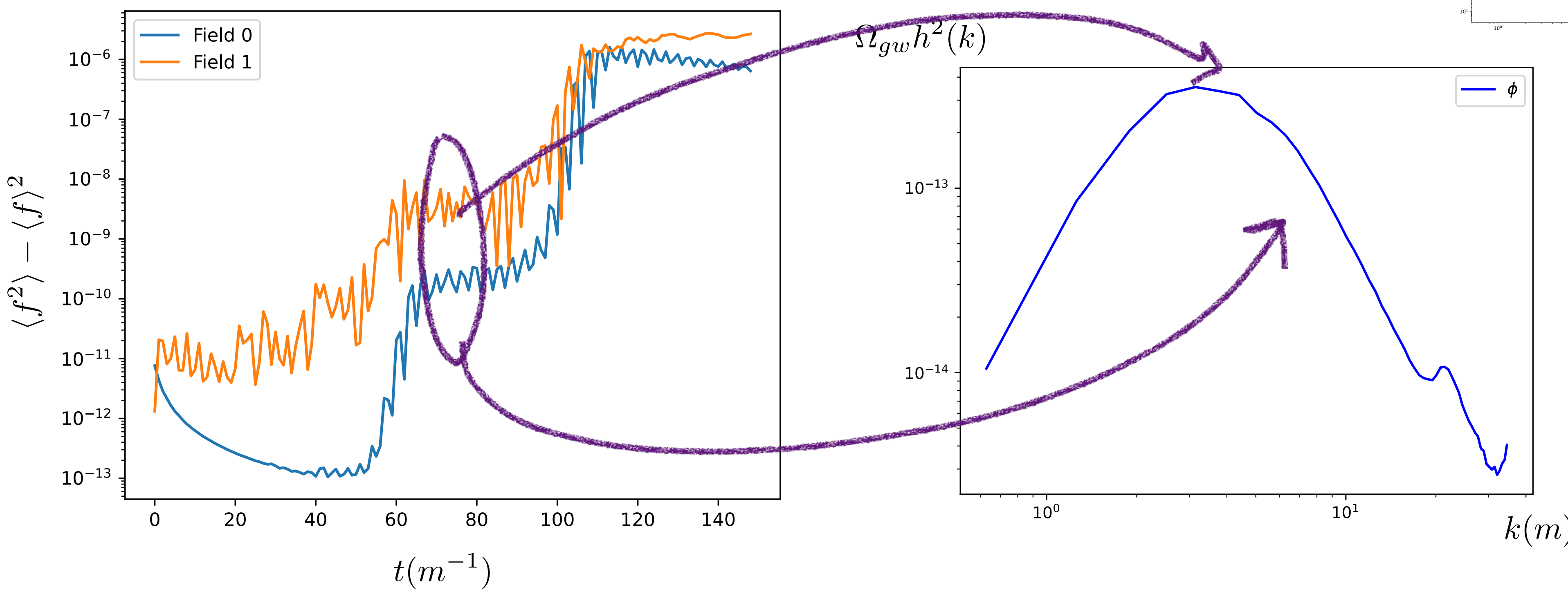

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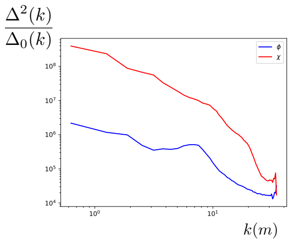
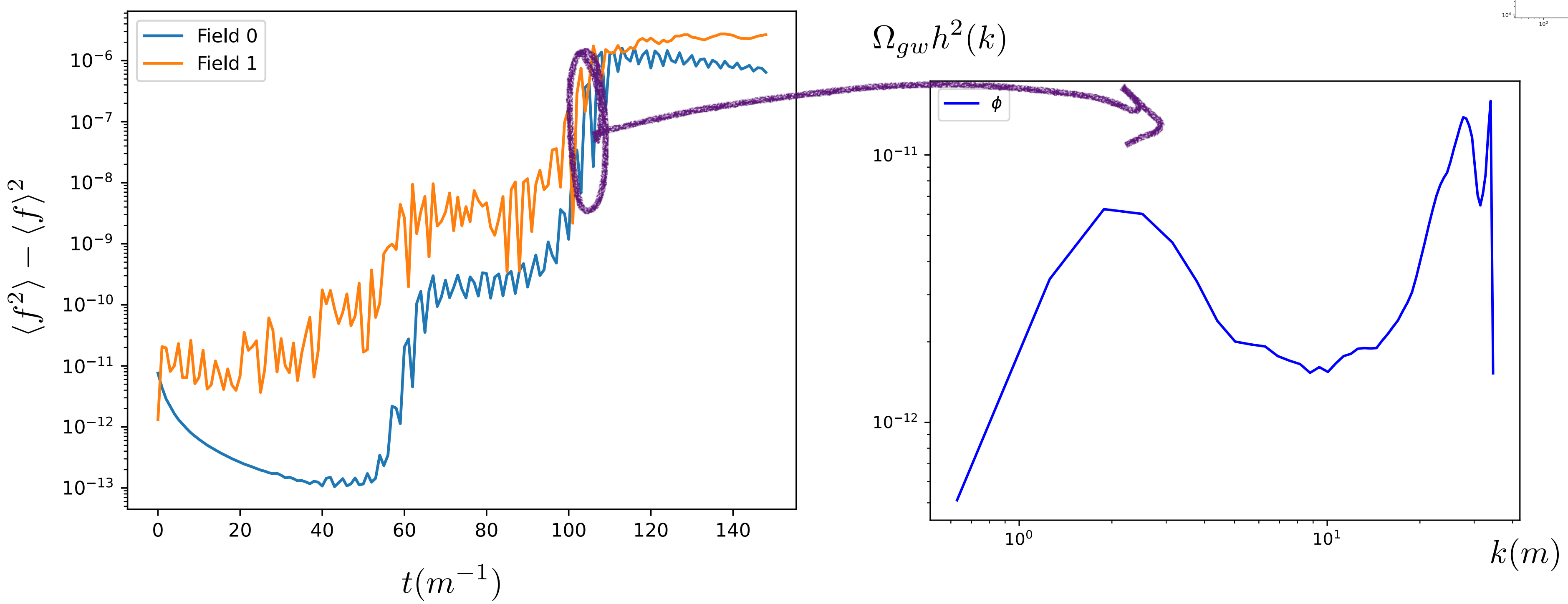
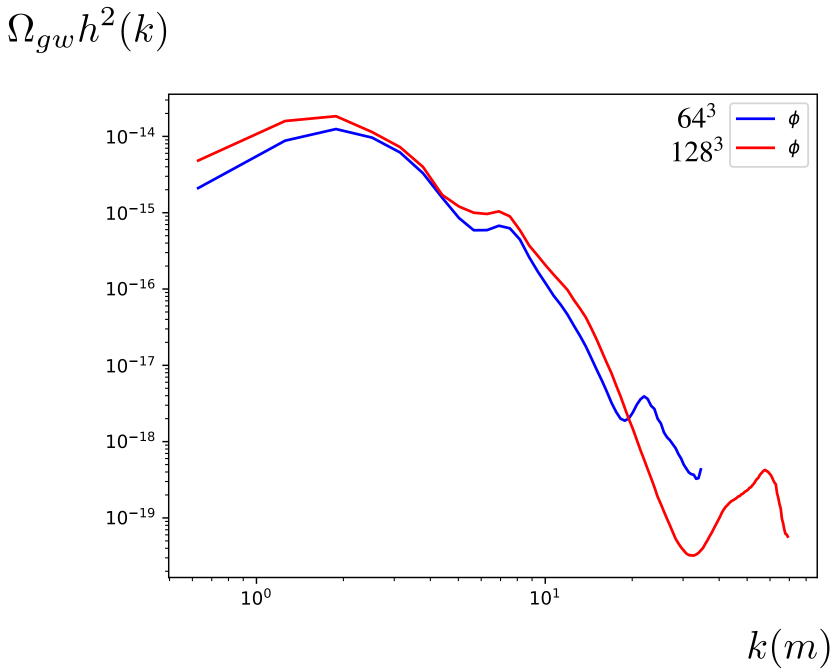


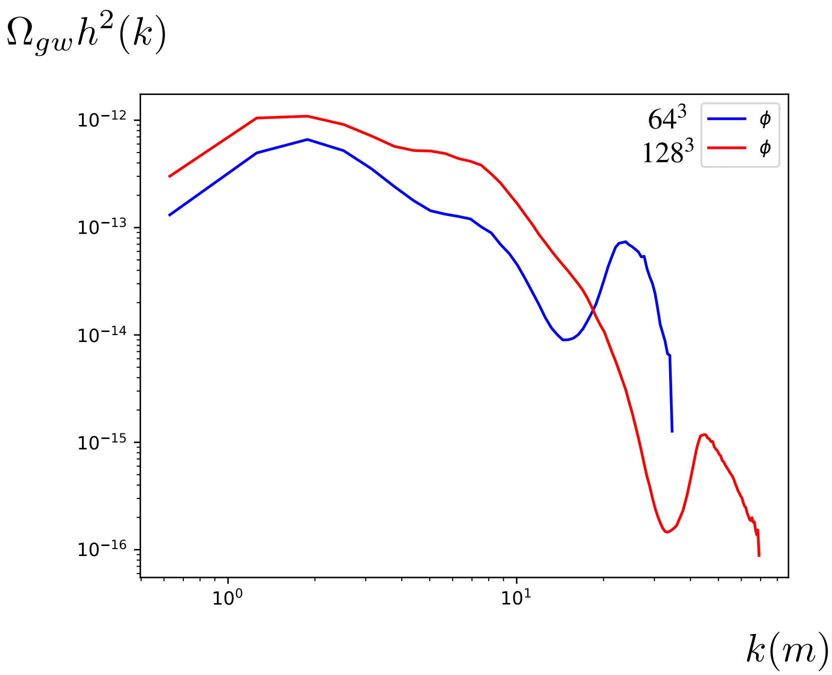



# How do we know what's real??

## These spectra *change* when you change the numerical parameters!




$t = 112\,m^{-1}$




$t = 117\,m^{-1}$

# Some things I find confusing

# Initial conditions

## The vacuum

- Assuming that modes are *sub-horizon* (or nearly sub-horizon, we assume that the fields have Bunch-Davies initial conditions,

$$\mathcal{P}_k = \frac{1}{2\omega_k} \qquad\qquad \omega_k = \sqrt{k^2 + m_{\text{eff}}^2}$$

- On the lattice, we need to translate this to the *discrete system.* For the most part this has a straightforward definition,

$$\phi(\vec{k}) = \frac{1}{(2\pi)^{3/2}} \int d^3x\, \phi(\vec{x}) e^{-i\vec{k}\cdot\vec{x}} \qquad\qquad \Phi(\vec{k}_j) = \frac{1}{(2\pi)^{3/2}} (\Delta x)^3 \sum_i \phi(\vec{x}_i) e^{-i\vec{k}_j\cdot\vec{x}_i}$$

# We *chose* the 2-pt correlation function

## to be the object that is invariant between the continuum and the discrete

$$\langle \phi(\vec{x})\phi(\vec{y})\rangle = \frac{1}{(2\pi)^3}\int d^3k\, d^3p\, \langle \phi(\vec{k})\phi(\vec{p})\rangle e^{i\vec{k}\cdot\vec{x}} e^{-i\vec{p}\cdot\vec{y}} = \frac{1}{(2\pi)^3}\int d^3k\, \mathcal{P}_k e^{i\vec{k}(\vec{x}-\vec{y})}$$

$$= \frac{1}{(2\pi)^3}(\Delta k)^6 \sum_i \sum_j \langle \Phi(\vec{k}_l)\Phi(\vec{p}_m)\rangle e^{i\vec{k}_l\cdot\vec{x}_i} e^{-i\vec{p}_m\cdot\vec{y}_j} = \frac{1}{(2\pi)^3}(\Delta k)^3 \sum_l \mathcal{P}_k e^{i\vec{k}_l(\vec{x}_i-\vec{y}_j)}$$

So there's a modification of the momentum-space 2-pt correlation function:

$$\langle \phi(\vec{k})\phi(\vec{p})\rangle = \mathcal{P}_k\, \delta(\vec{k}-\vec{p})$$

$$\langle \Phi(\vec{k}_l)\Phi(\vec{p}_m)\rangle = (\Delta k)^3\, \mathcal{P}_k\, \delta_{lm} = \left(\frac{L}{2\pi}\right)^3 \mathcal{P}_k\, \delta_{lm}$$

# This has an impact on the initial conditions

- When we convert the power spectrum to *dimensionless* units,

$$\langle|\phi_{\rm pr}(k_j)|\rangle = \left(\frac{B^6}{m_{\rm pl}^2}\right)\left(\frac{L_{\rm pr}}{B^2 2\pi}\right)^3 \mathcal{P}_k \qquad \mathcal{P}_k^{\rm pr} = \frac{B^3}{m_{\rm pl}^2}\mathcal{P}_k$$

- Which means for Bunch-Davies….

$$\mathcal{P}_{k,\rm BD}^{\rm pr} = \frac{B^3}{m_{\rm pl}^2}\mathcal{P}_{k,\rm BD} = \frac{B^2}{m_{\rm pl}^2}\frac{1}{2\omega_{\rm pr}}$$

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!

# Thoughts on strong hyperbolicity

## …make it wavy

- The tricks we play (as theorists) to reduce the number of degrees of freedom can negatively affect numerical stability
- Adding degrees of freedom keep equations strongly hyperbolic (that is, wave-like) which means you need to store (and evolve) more information than you *have to*.
- But it makes the problem solvable. Examples include:
  - Using Lorenz gauge, and keep track of constraints
  - Yesterday, I talked about BSSN and how this works for Numerical Relativity
  - Stiff equations of motion can be stabilized with extra degrees of freedom

# Looking at Scalar Galileons

## We can

- We start with a stiff, derivatively coupled equation of motion,

$$\Box\pi + \frac{1}{3\Lambda^3}\left((\Box\pi)^2 - (\partial_\mu\partial_\nu\pi)^2\right) = -\frac{T}{3\,m_{\rm pl}}$$

- By identifying

$$H_{\mu\nu} = \partial_\mu\partial_\nu\pi \qquad\qquad A_\mu = \partial_\mu\pi$$

- We get a more complicated system, but one that's strongly hyperbolic

$$\Box\pi + \frac{1}{3\Lambda^3}\left(H^{\mu\nu}H_{\mu\nu} - (H^\nu_\nu)^2\right) = -\frac{T}{3\,m_{\rm pl}}$$

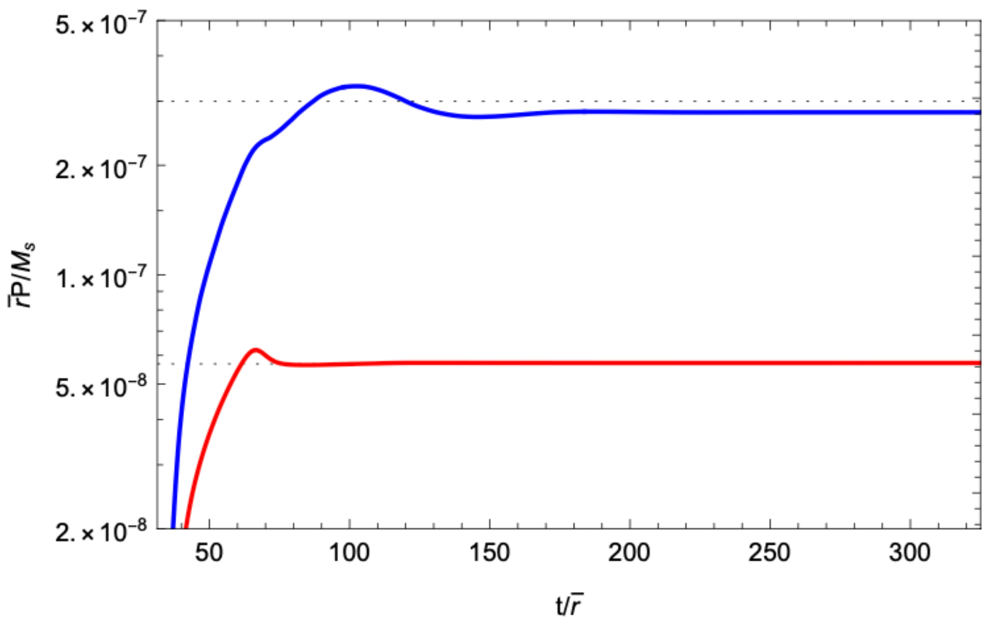
$$\Box A_\mu - \frac{1}{\tau}\partial_t A_\mu - M^2 A_\mu = -M^2\partial_\mu\pi$$

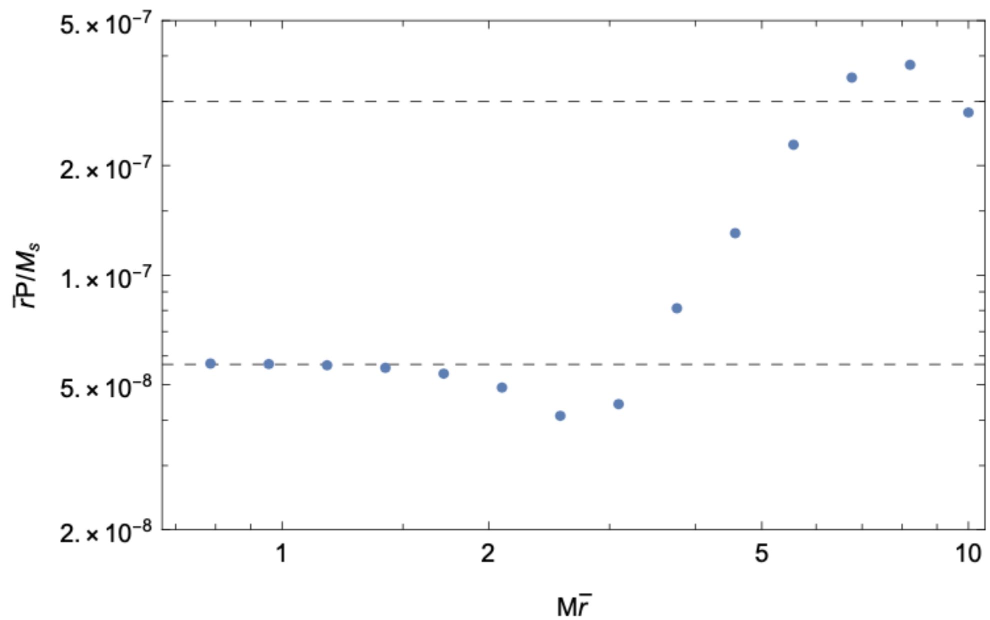
$$\Box H_{\mu\nu} - \frac{1}{\tau}\partial_t H_{\mu\nu} - M^2 H_{\mu\nu} = -\frac{M^2}{2}\left(\partial_\mu A_\mu + \partial_\nu A_\mu\right)$$

2205.05697

# We get the behavior of the full system

## …but much more reliably



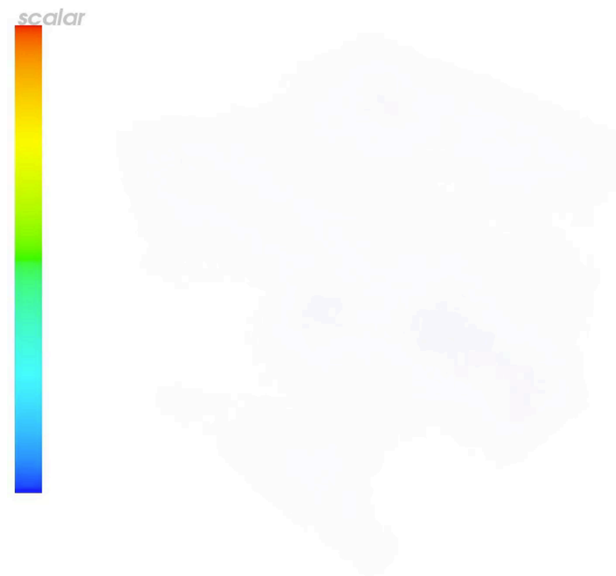




2205.05697

# A quick example
## …in the case of oscillons

- Many potentials create oscillons; we've been interested in $\alpha$ -attractor models which, in the absence of a coupled field, are great oscillon producers
- An open question has been, *do oscillons decay* which can be studied by considering an explicit coupling to another field

$$\mathcal{L}_{\text{int}} = \frac{g^2}{2}\phi^2\chi^2$$




Peter Krosniak
'27

J'sun Gardner
'26

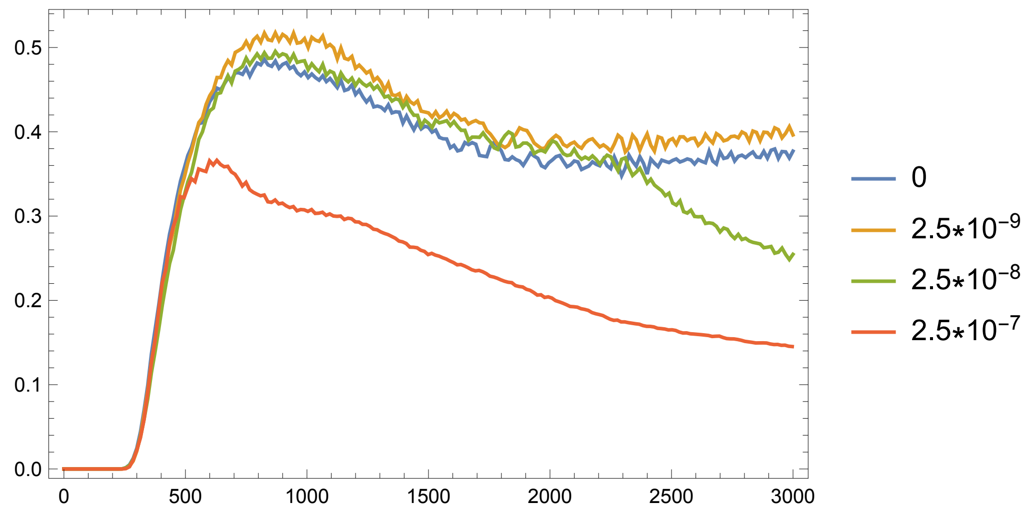
# The End of the Oscillon

- Consider the E-model $\alpha$-attractor

$$V = \frac{m^2\mu^2}{2}\left(1 - e^{-\frac{\phi}{\mu}}\right)^2$$

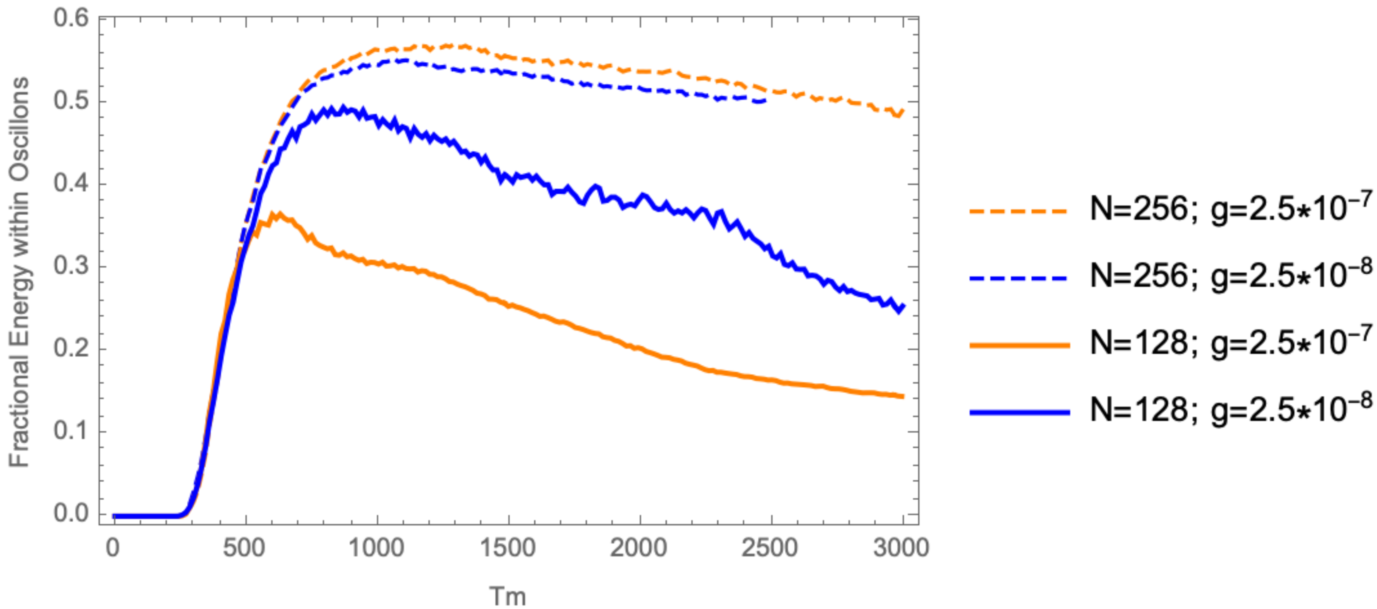
- We can see that the energy in oscillons decays parametrically with the strength of the interaction

- But is it *real?*



# When we look at

## …the convergence test

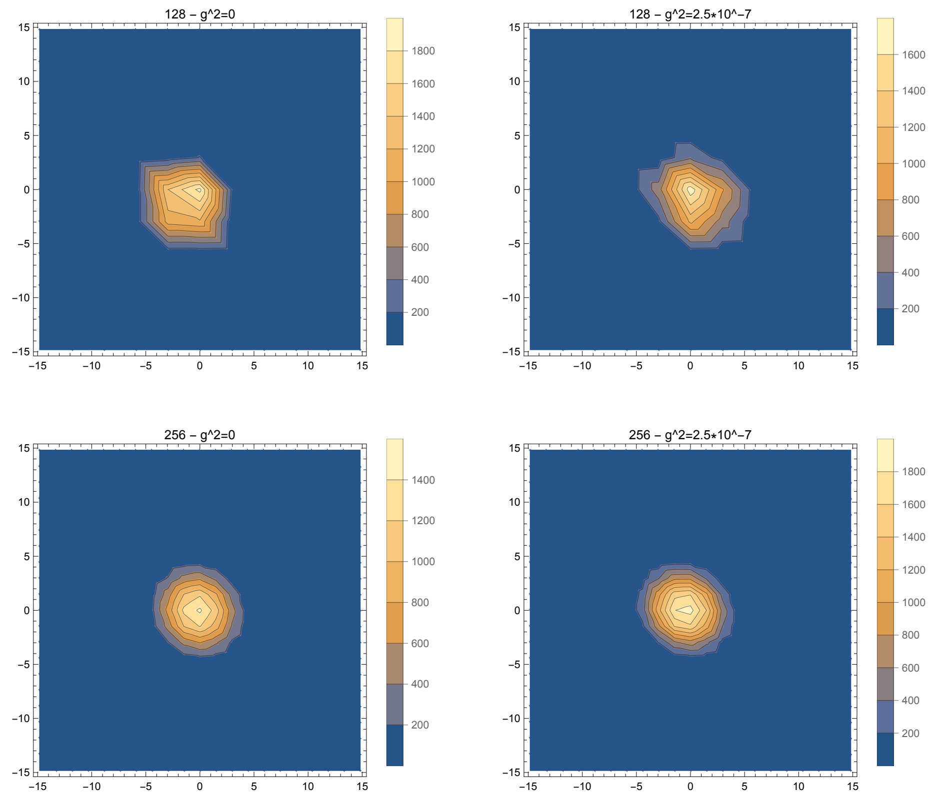


# Why?

## …we can actually see what's going on!

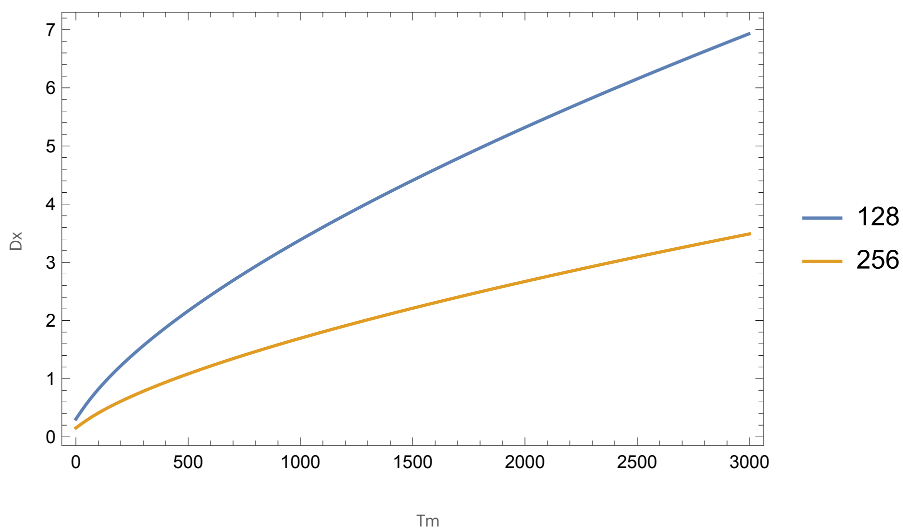
The size and shape of oscillons in 128^3 and 256^3 resolutions are different

128^3 isn't enough points to form oscillons with a circular shape

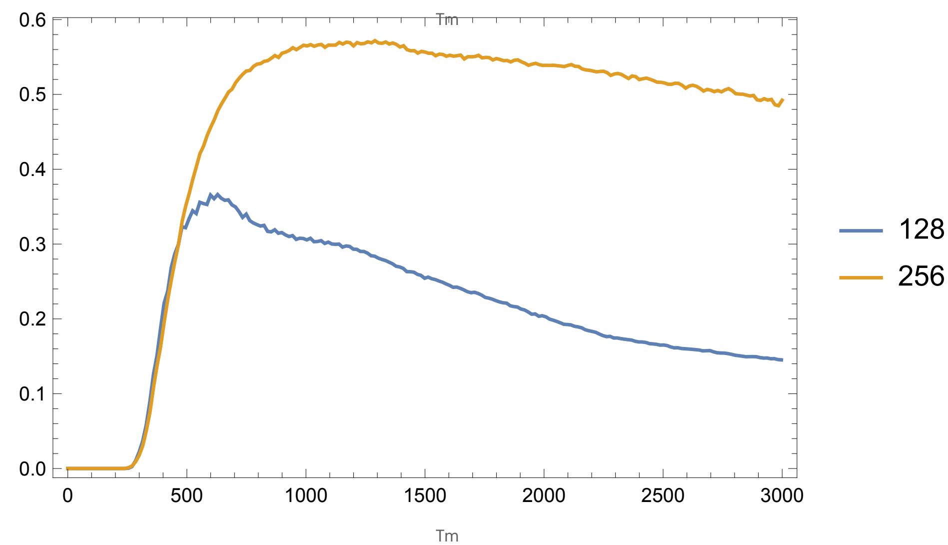


# When do we approach the modes
## …that we can't trust?

Physical dx of E-Model 128 and 256 2.5*10^-7 runs vs Time




E-Model 128 and 256 2.5*10^-7 runs vs Time



# For this afternoon:

## … if you're participating

- Please download GABE
  - cosmo.kenyon.edu/gabe.html
- If you have a MAC
  - Try to run the instructions on that page, new(ish) operating systems should work without issue
- If you have LINUX
  - Please install fftw3 (enabling regular and long-double: openmp and threads for each)
- If you plan to use the remote version, please email me (giblinj@kenyon.edu) as soon as possible!
- **Please also have Jupyter (or similar) for plotting. I will distribute a .ipynb this afternoon!**

# Advice from my students

- Write the code yourself, so you can change anything. Seems silly but it was helpful to me to realize that I can communicate via the code (printf("I'm here :)\n");)
- When in doubt, plot it out - there's lots of data and it can tell you many things
- Write in *words* what you want the code to do. Translate it to a step by step list and check that this matches the order of the code
- As always - fix the bug you know :)
- The physics doesn't care about the math and we want to find the region where it doesn't care about the code either, so change code parameters to find the physics
- This is a basic idea but the code is only as smart as you let it be, so let it be dumb and change single things at a time so you know for sure what is going on.
- Go to a known regime: homogeneous, static, symmetric. Check it behaves as expected