Assessment of universal relations among second-order moments of relativistic stars via reformulated perturbation equations

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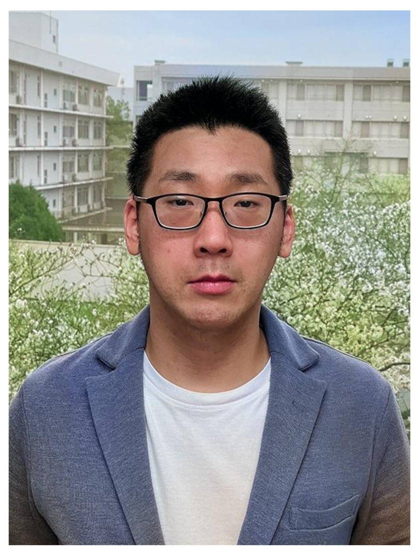
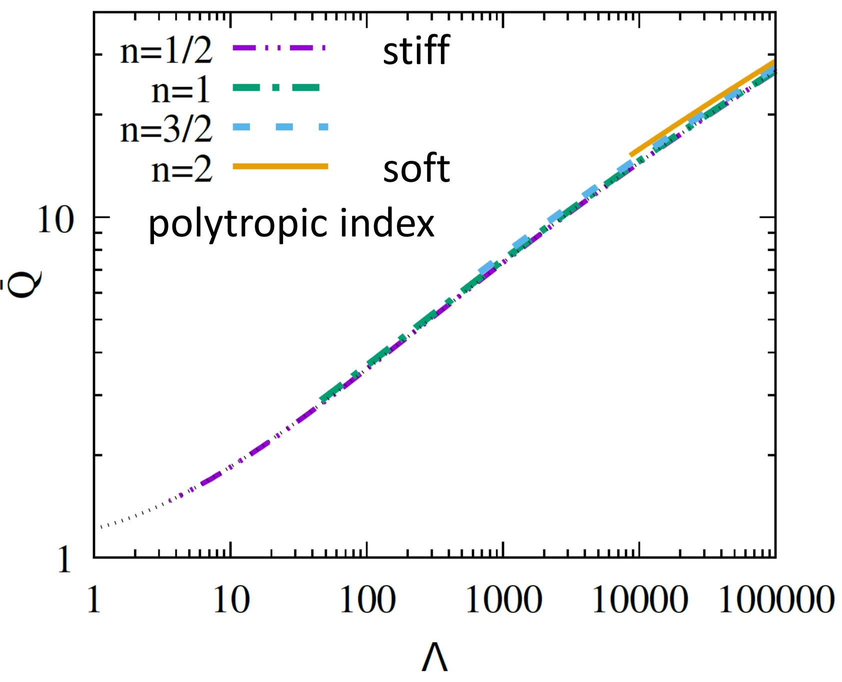


Matter effect in gravitational waves from binary neutron stars

Tidal deformability Λ: how the star is deformed due to the tidal field of the companion
Spin-induced quadrupole moment Q: how the star is deformed due to its own rotation

Inferring both quantities introduce parameter correlation and degrade the accuracy
Fortunately, $Q \approx Q(\Lambda)$ holds for neutron-star-like equations of state --- so-called I-Love-Q
This is adopted in most gravitational-wave data analysis



But this relation is sometimes adopted even when the equations of state are sampled directly [LIGO&Virgo (2018)]

Some appear to think that the computation of Q more complicated than it actually is

Question: How many ODEs do we need to solve for deriving I, Λ, and Q ?

Naïve answer: 11, and we need to combine particular and homogeneous solutions appropriately to obtain Q

Our proposal: 6 with no matching needed, and we may be able to identify variables governing each moment

(0) TOV star $\left(ds^2 = -e^{2\nu}dt^2 + e^{2\lambda}dr^2 + r^2 d\Omega\right)$

$$\frac{dP}{dr} = -\frac{(e+P)(m+4\pi P r^3)}{r(r-2m)}$$

$$\frac{dm}{dr} = 4\pi\rho r^2$$

$$\cancel{\frac{d\nu}{dr} = \frac{m+4\pi P r^3}{r(r-2m)}}$$

Relativistic Bernoulli' theorem gives the metric

$$\nu = \ln\left(1 - \frac{2M}{r}\right) - H, \qquad H := \int \frac{dP}{e(P)+P}$$

We may also change independent variable from r to for H for handling the surface $H = 0$ (see, e.g., Lindblom 1992 for discussions)

(1) Moment of inertia I $\left(j := e^{\nu+\lambda}\right)$

$$\cancel{\frac{1}{r^4}\frac{d}{dr}\left(r^4 j \frac{d\bar{\omega}}{dr}\right) + \left[\frac{4}{r}\frac{dj}{dr} - e^{\lambda-\nu}\frac{l(l+1)-2}{r}\right]\bar{\omega} = 0}$$

We solve for $w := d\ln\bar{\omega}/d\ln r$ by (Zdunik+ 1987)

$$\frac{dw}{dr} = -\frac{w(w+3)}{r} + \frac{4\pi(w+4)(e+P)r^2}{r-2m}$$

Similar for tidal deformability Λ (Postnikov+ 2010)

(2) Spin-induced quadrupole moment Q

Traditional approach: We schematically solve

$$\frac{d}{dr}\begin{pmatrix} h \\ v \end{pmatrix} = \begin{pmatrix} L_{hh} & L_{hv} \\ L_{vh} & 0 \end{pmatrix}\begin{pmatrix} h \\ v \end{pmatrix} + \begin{pmatrix} W_h \\ W_v \end{pmatrix}$$

Once with sources W_h and W_v, nonlinear in $\bar{\omega}$ and $d\bar{\omega}/dr$, for particular solution
Once with $W_h = 0 = W_v$ to obtain homogenous solutions
The two sets of solutions are combined to ensure regularity at the stellar surface and then we can extract information of the quadrupole moment … a bit messy
But they cannot be combined to a master equation because $L_{hv} \neq$ const. in the star

Our proposal: We force the above matrix to be "diagonalized" by introducing

$$q(r) := v(r) + f(r)h(r)$$

Along with determining the auxiliary function f by

$$\frac{df}{dr} = -\frac{2}{m}f^2 + \left\{2\frac{d\nu}{dr} - \frac{r^2}{m+4\pi P r^3}\left[4\pi(e+P) - \frac{2m}{r^3}\right]\right\}f + 2\frac{d\nu}{dr}$$

we need to solve the following equation only once to obtain the quadrupole moment

$$\frac{d\tilde{q}}{dr} = -\left[\frac{2f}{m+4\pi P r^3} + \frac{2(w+1)}{r}\right]\tilde{q} + \frac{j^2 w^2}{6}\left\{\frac{d\nu}{dr} + \frac{1}{r} + \left[\frac{d\nu}{dr} - \frac{1}{2(m+4\pi P r^3)}\right]f\right\}$$
$$+ \frac{8\pi(e+P)r^3 j^2}{3(r-2m)}\left\{\frac{d\nu}{dr} + \frac{1}{r} + \left[\frac{d\nu}{dr} + \frac{1}{2(m+4\pi P r^3)}\right]f\right\}, \qquad \tilde{q} := \frac{q}{(\bar{\omega}r)^2}$$

I and Q are determined by surface values of w and $\{f, \tilde{q}\}$, respectively (see the paper)

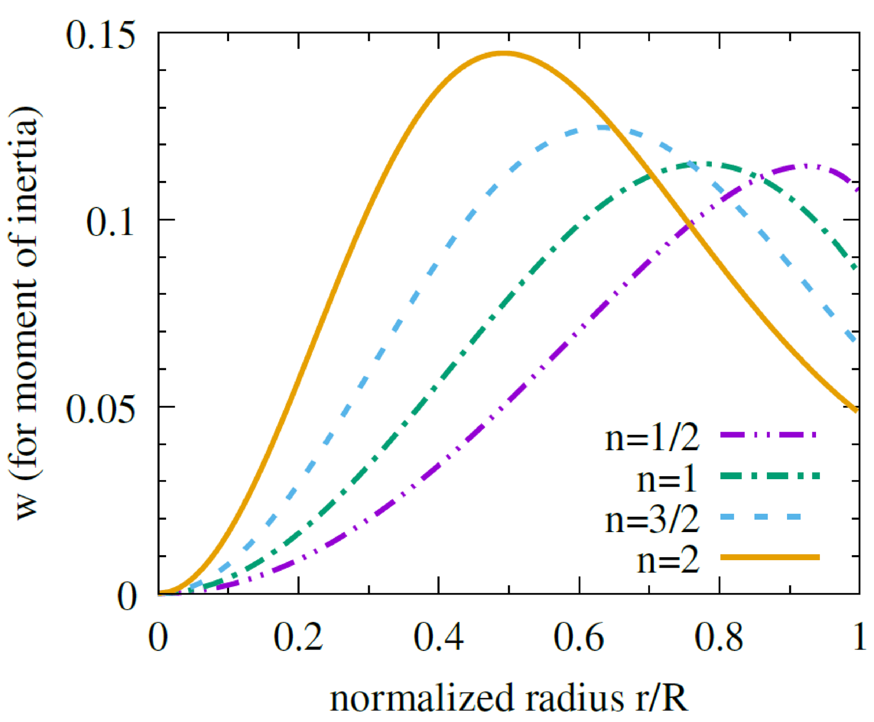
Application: assessment of universal, I-Love-Q relation

It is known that the "universality" tends to be achieved for (1) stiff equations of state and/or (2) high compactness

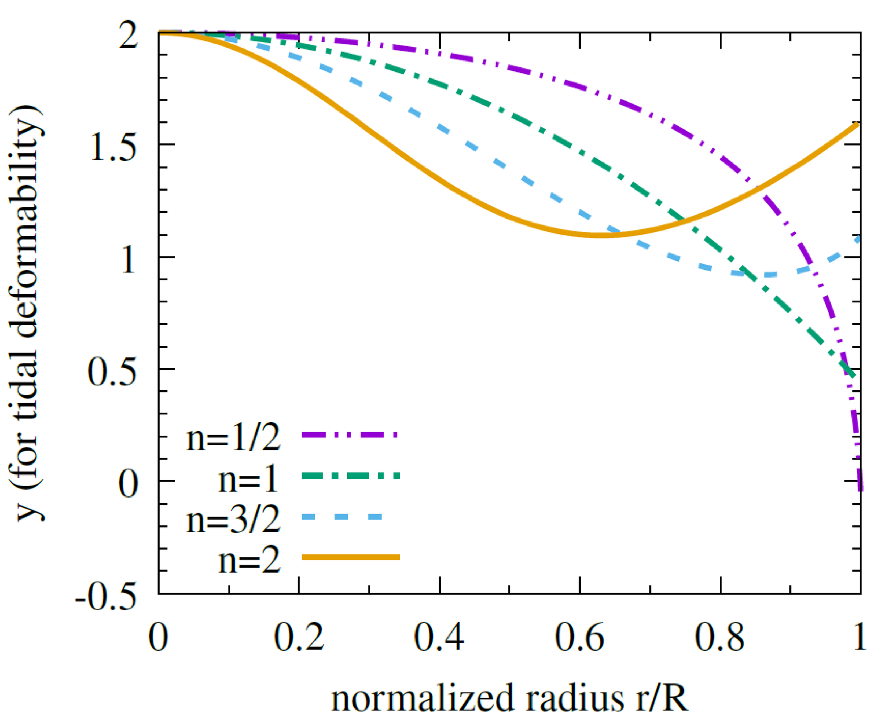
Profile of the variables at $C = GM/Rc^2 = 0.05$ (not very compact)

For soft equations of state, i.e., large values of n, profiles are characterized by short-scale variations

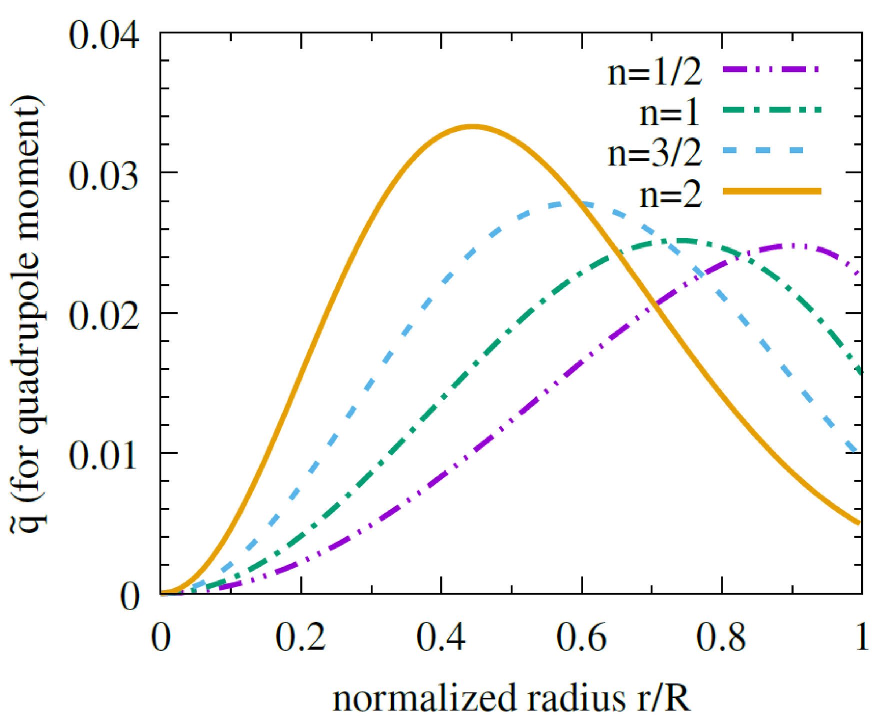
moment of inertia (I)



tidal deformability (Love)



quadrupole moment (Q)



Conversely, universality is realized when the variation length scale is comparable or longer than the stellar radius
cf. other universality <-> correlation length in critical phenomena, scattering length in few-body quantum systems, …
[But recall the universality for (2) is enforced by the black-hole limit despite short-scale variations for all n's]